MAT 1225 Calculus of a Single Variable I

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Contents

0 Prerequisites

0.1 Review of Functions

0.1.1 The Definition

Definition. A *function* f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. We write $f : A \to B$ to formally represent the above.

The set A above is called the *domain* and the set B is called the *codomain*, if **every** element in B can be written as $f(x)$ for some x, then we call B the *range*. There are many important terms associated with functions:

- **Independent Variable**: associated with the domain of a function, i.e. the x variable.
- **Dependent Variable**: associated with the range of a function, i.e. the $f(x)$'s.
- **Graph of a Function**: the set of all points of the form $(x, f(x))$ where x varies throughout the entire domain.
- **Argument of a Function**: the expression on which the function is evaluated.

  For example: x is the argument of $f(x)$; 7 is the argument of $f(7)$; $x^5 - 45$ is the argument of $f(x^5 - 45)$

0.1.2 Catalog of Essential Functions

1. **Polynomials**: are functions of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

   - The a_i's are called the *coefficients* of the polynomial.
   - The number n is called the *degree* of the polynomial.

2. **Rational Functions**: are functions of the form $\dfrac{p(x)}{q(x)}$ where p and q are polynomials.

   For example: $\dfrac{5x^3 - 13}{2x^2 - x + 5}$

3. **Algebraic Functions**: are functions constructed using algebraic operations

   For example: $f(x) = \sqrt{x^5 - 7x + 5}$; $g(x) = x^{1/7}(x^2 - 2)$

4. **Exponential Functions**: have the form $f(x) = b^a$, where $b \neq 1$ is a positive real number. Logarithmic functions go hand-in-hand with these. For the following important rules of exponential functions, let $b \neq 1$ be a positive real number.

   - $b^x b^y = b^{x+y}$ for all real numbers x and y
   - $(b^x)^y = b^{xy}$

5. **Logarithmic Functions**: have the form $f(x) = \log_b(x)$, where $b \neq 1$ is a positive real number. These are related to exponential functions in the following way1

$$x = b^y \quad \text{is equivalent to} \quad y = \log_b(x).$$

   For the following important rules follow from the rules of exponential functions.

- $\log_b(xy) = \log_b(x) + \log_b(y)$ for all positive x and y
- $\log_b(x^y) = y \log_b(x)$ for all real numbers y and positive real numbers x

6. **Trigonometric Functions**: $\sin(x), \cos(x), \tan(x), \csc(x), \sec(x), \cot(x)$. These are fundamental to many branches of mathematics and engineering.
7. **Piece-wise Functions**: As the name suggests, these are functions comprised of pieces of other functions. For example:

$$f(x) = \begin{cases} \sin(x) & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ e^{x-1} & \text{if } x > 1 \end{cases}$$

0.1.3 Transformations of Functions

Shifts/Translations: let $c > 0$

1. $f(x) + c$ shifts the function f up by c
2. $f(x) - c$ shifts the function f down by c
3. $f(x + c)$ shifts the function f to the left by c
4. $f(x - c)$ shifts the function f to the right by c

Stretches and Reflections: let $c > 1$

1. $cf(x)$ stretches f vertically by a factor of c
2. $\frac{1}{c}f(x)$ compresses f vertically by a factor of c
3. $f(cx)$ compresses f horizontally by a factor of c
4. $f(\frac{1}{c}x)$ stretches f horizontally by a factor of c
5. $-f(x)$ reflects f about the x-axis
6. $f(-x)$ reflects f about the y-axis

Combinations of Functions:

1. $(f + g)(x) = f(x) + g(x)$
2. $(f - g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, on the proper domain
5. ***IMPORTANT!*** $(f \circ g)(x) = f(g(x))$, a composition of functions

0.1.4 Inverse Functions

Definition. Let $f : A \to B$ be a function. If there exists a function $g : B \to A$ so that $f \circ g : B \to B$ is the identity function for B **and** $g \circ f : A \to A$ is the identity function for A, we call g the inverse function of f and denote it f^{-1}.

There may not always be an inverse function for any given f. This brings up the need for the following definitions.

Definition. A function $f : A \to B$ is called *one-to-one* if for every element b in the set B, there is **at most** one element a in the set A such that $f(a) = b$.

Definition. A function $f : A \to B$ is called *onto* if for every element b in the set B, there is **at least** one element a in the set A such that $f(a) = b$.

Proposition 0.1.1. *A function $f : A \to B$ has an inverse function if and only if f is both one-to-one and onto.*

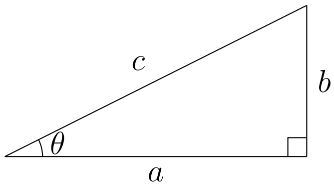
If a function does not have an inverse, not all is lost. The trick is to find an interval where f is both one-to-one and onto, then just pretend that the restricted domain and range were the original ones.

Basic idea for finding inverses of a function f:

1. Find an interval where f is one-to-one and onto.
2. Replace $f(x)$ with a simpler symbol (might I suggest the letter y?).
3. Switch the roles of x and y in the equation.
4. Solve the above equation for y.
5. Replace the symbol y with $f^{-1}(x)$.

0.2 Trigonometric Identities

Trigonometric Functions



From the right triangle pictured above, we have the following function definitions

$$\sin(\theta) = \frac{b}{c} \qquad \cos(\theta) = \frac{a}{c} \qquad \tan(\theta) = \frac{b}{a}$$

$$\csc(\theta) = \frac{c}{b} \qquad \sec(\theta) = \frac{c}{a} \qquad \cot(\theta) = \frac{a}{b}$$

Angle Sum/Difference Formulas

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Double-Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Power Reducing Formulas

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$
$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Half-Angle Formulas

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$$
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$$
$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$$

Product-to-Sum Formulas

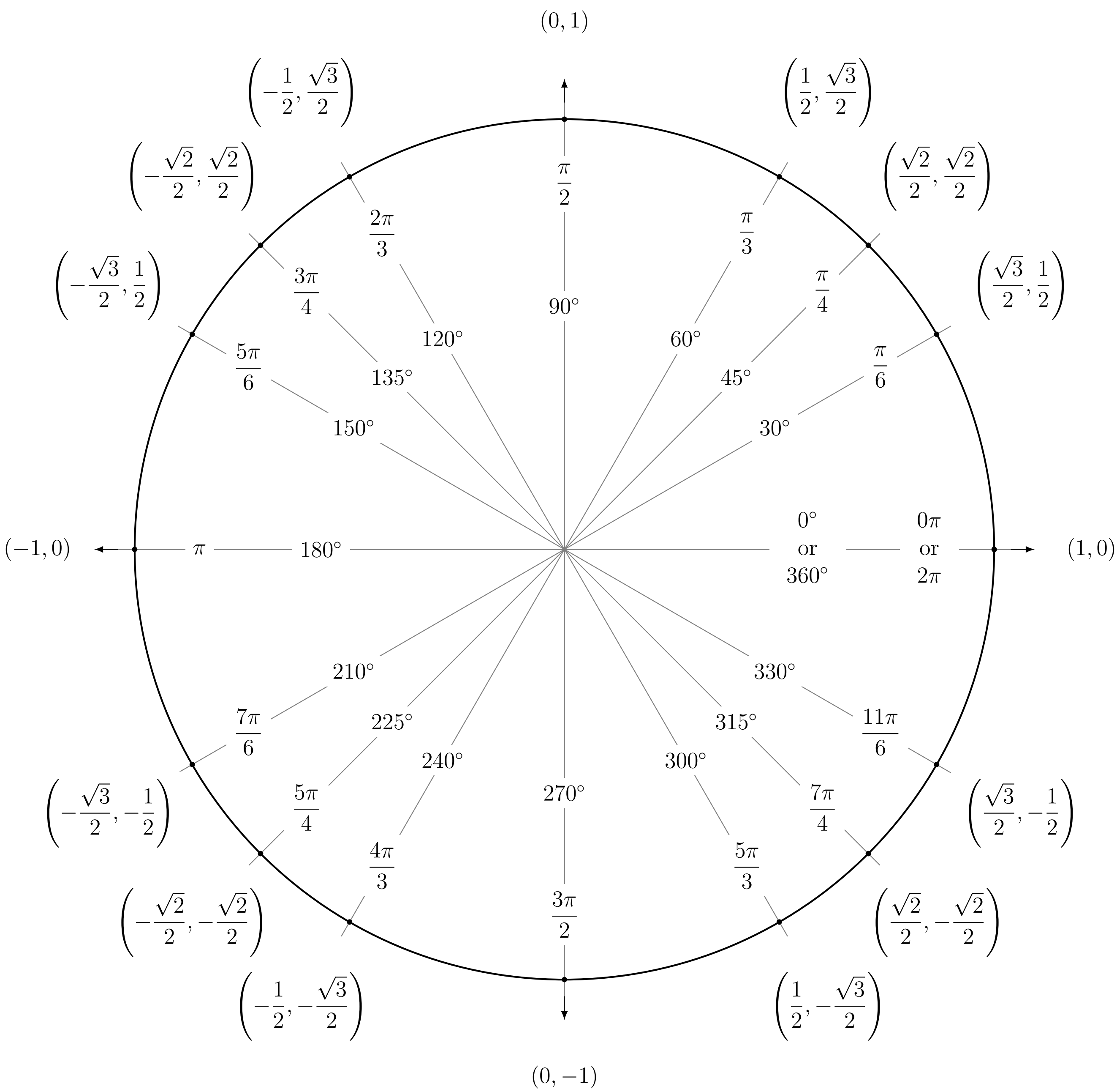
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$$
$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$$

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$
$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

0.2.1 The Unit Circle

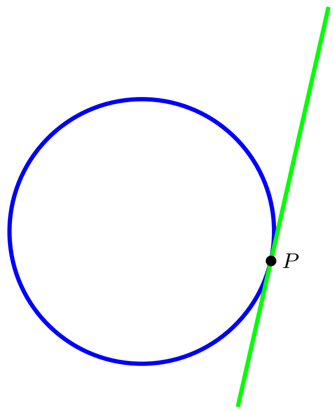
Points on the unit circle are given by $(x, y) = (\cos\theta, \sin\theta)$. The most important angles to know are listed below, along with the relevant coordinates on the unit circle. To remember this most efficiently, it really suffices just to remember the first quadrant as there is plenty of symmetry.



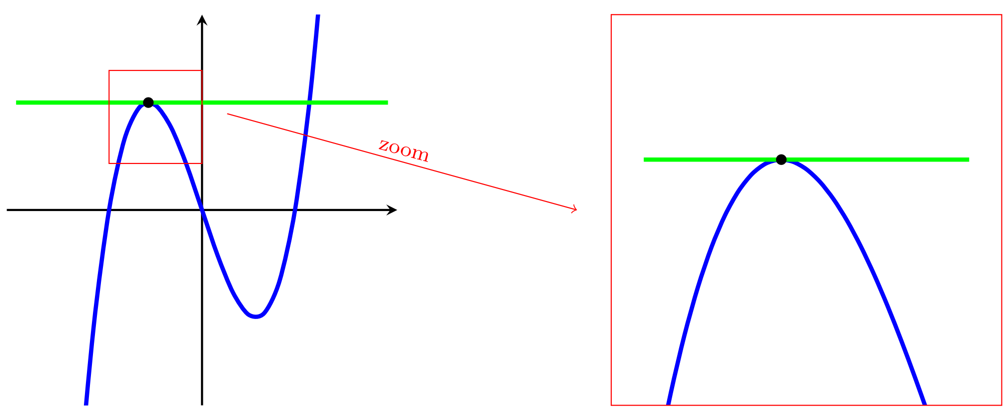
2 Limits and Derivatives

2.1 The Tangent and Velocity Problems

You probably have a notion of a tangent line from a previous course in geometry.



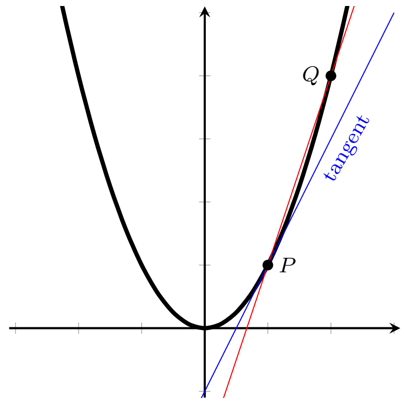
A line is tangent to a circle if it touches the circle in exactly one point. More generally, a line is tangent to a curve if it intersects that curve in exactly one point and (if we zoom in close enough to that point) the line sits entirely on one side of the curve.



Example 2.1.1. Find the equation for the tangent line to the curve $y = x^2$ through the point $P = (1, 1)$.

Since we already know a point on the line, all we need to do is figure out the slope of the tangent line. This, however, is problematic - we need to know two points on the line to compute a slope, but we only know of $P = (1, 1)$.

We can approximate the slope, however, by computing the slope of the **secant line** through both $P = (1, 1)$ and $Q = (x, x^2)$ for varying values of x.



As (x, x^2) gets closer to $(1, 1)$, the secant line gets closer to the tangent line. Let's compute some slopes for varying values of $Q = (x, x^2)$. Note that when $x \neq 1$, the slope is given by $m_{PQ} = \dfrac{x^2 - 1}{x - 1} = x + 1$.

x	$m_{PQ} = x + 1$
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

From this table, the slope of the tangent line seems to be limiting to 2 and thus the equation of the tangent line through $(1, 1)$ is given by $y = 2x - 1$.

Example 2.1.2. A ball is thrown vertically upward from a height of 48 ft and with an initial velicity of 32ft/s. The ball's height above the ground at time t is given by $h(t) = -16t^2 + 32t + 48$. What is the ball's (instantaneous) velocity 1 s after the ball is thrown?

Recall that

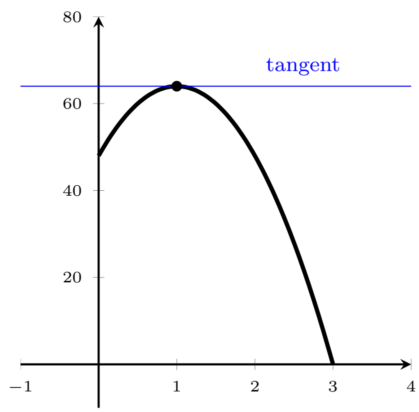
$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{h(t_1) - h(t_0)}{t_1 - t_0},$$

which is exactly the slope of the secant line through the points $(t_0, h(t_0))$ and $(t_1, h(t_1))$. We'll approach as in the previous example and make a table of these slope values to approximate the instantaneous velocity at $t = 1$. To simplify computations:

$$m_t = \frac{h(t) - h(1)}{t - 1} = \frac{[-16t^2 + 32t + 48] - [64]}{t - 1} = \frac{-16t^2 + 32t - 16}{t - 1} = \frac{-16(t^2 - 2t + 1)}{t - 1} = \frac{-16(t - 1)(t - 3)}{t - 1} = -$$

t	$m_t = -16(t-1)$
0	$16\,\text{ft/s}$
0.5	$8\,\text{ft/s}$
0.9	$1.6\,\text{ft/s}$
0.99	$0.16\,\text{ft/s}$
0.999	$0.016\,\text{ft/s}$

These numbers are limiting to an instantaneous velocity of $0,\text{ft/s}$, and this physically makes sense: at $t = 1\,\text{s}$, the ball as reached the apex and is about to start traveling downard.



Both of these examples hint at the core idea of calculus - limits of finer and finer approximations. In subsequent sections, we will formalize these notions of limits and approximations.

2.2 The Limit of a Function

Definition. Let f be a function and suppose that $f(x)$ is defined for all x very near the number a. If we can pick x-values so that $f(x)$ is arbitrarily close to some number L when x sufficiently close to a (when both $x < a$ and $x > a$), then we write

$$\lim_{x \to a} f(x) = L$$

and say "the **limit** of $f(x)$, as x approaches a, is L."

Remark. Notice that the definition does not require that $f(x)$ be defined when $x = a$. Notice that the limit also requires that we can approach from either side of a to get the same L value.

Example 2.2.1. Using a table of values, guess the limit of the function $f(x) = \dfrac{x^2 - 16}{x + 4}$ as $x \to -4$.

$x < -4$	$f(x)$
-4.1	-8.1
-4.01	-8.01
-4.001	-8.001
-4.0001	-8.0001
-4.00001	-8.00001

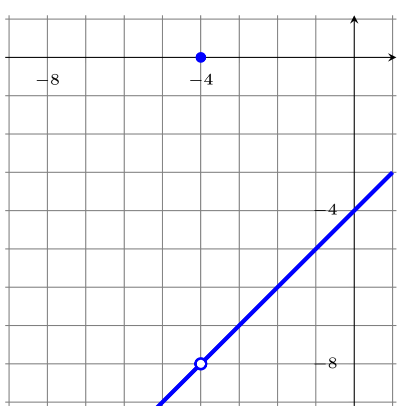
$x > -4$	$f(x)$
-3.9	-7.9
-3.99	-7.99
-3.999	-7.999
-3.9999	-7.9999
-3.99999	-7.99999

Limit: -8

Example 2.2.2. Let g be the function given by

$$g(x) = \begin{cases} \dfrac{x^2 - 16}{x + 4} & \text{if } x \neq -4, \\ 0 & \text{if } x = -4. \end{cases}$$

Use a graph to determine $\lim_{x \to -4} g(x)$.



Limit: -8

Remark. This last example demonstrates that, even if $f(a)$ is defined, $\lim_{x\to a} f(x)$ is independent of the function's value at a.

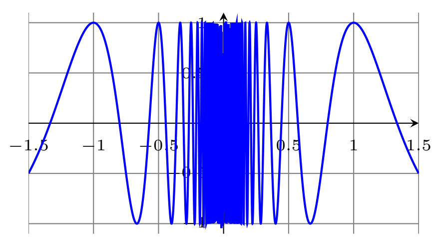
Example 2.2.3. Consider the function $f(x) = \cos\left(\frac{\pi}{2x}\right)$. Using a table of values, guess the limit $\lim\limits_{x\to 0} f(x)$.

$x < 0$	$f(x)$
-0.05	1
-0.01	1
-0.005	1
-0.001	1
-0.0005	1

$x > 0$	$f(x)$
0.05	1
0.01	1
0.005	1
0.001	1
0.0005	1

Limit: 1?

Now look at the graph of f below. Notice that the output doesn't just settle on 1, but rather oscillates rapidly at $x \to 0$. Since $f(x)$ never actually settles on a number at all (it hits every value between -1 and 1 infinitely many times), we have that the limit does not exist. This is one of the pitfalls that we can run into if we just use the calculator to guess and check what limits may be.



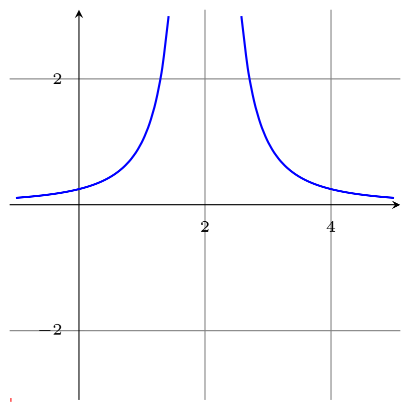
Example 2.2.4. Using a table of values, determine the limit $\lim\limits_{x\to 2} \frac{1}{(x-2)^2}$, if it exists.

$x < 2$	$f(x)$
1	1
1.9	100
1.99	10^4
1.999	10^6
1.9999	10^8

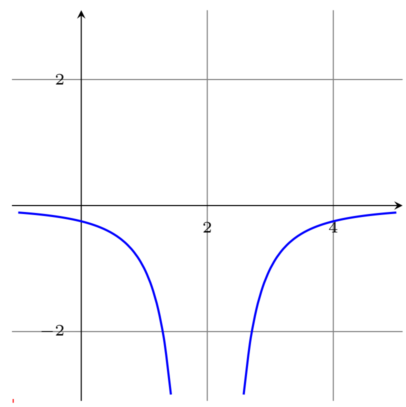
$x > 2$	$f(x)$
3	1
2.1	100
2.01	10^4
2.001	10^4
2.0001	10^8

Limit: Does Not Exist

2.2.1 Infinite Limits



Graph of $y = \frac{1}{(x-2)^2}$



Graph of $y = \frac{-1}{(x-2)^2}$

We saw in the previous example that

$$\lim_{x \to 2} \frac{1}{(x-2)^2}$$

did not exist, as the function grew unbounded as $x \to 2$. Similarly,

$$\lim_{x \to 2} \frac{-1}{(x-2)^2}$$

does not exist, but the graph is different - rather than growing positively unbounded, this functions becomes negatively unbounded. Just saying that a limit "does not exist" does not really capture the behavior of the graph. So, to emphasize the difference, we write

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty \quad \text{and} \quad \lim_{x \to 2} \frac{-1}{(x-2)^2} = -\infty.$$

Definition. Let f be a function defined on both sides of a, except possibly at a. If $f(x)$ becomes positively unbounded as x approaches a, then we write

$$\lim_{x \to a} f(x) = \infty.$$

Similarly, if $f(x)$ becomes negatively unbounded as x approaches a, then we write

$$\lim_{x \to a} f(x) = -\infty.$$

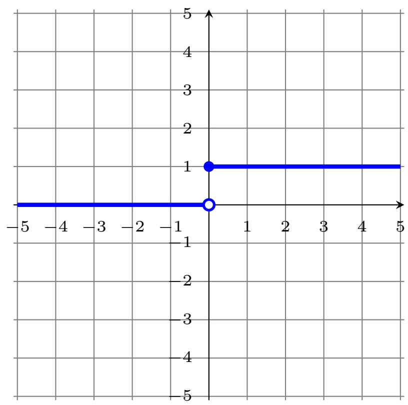
Remark. ∞ is *not a real number*, so the above limit *does not exist.* However, we this is convenient notation.

2.2.2 One-Sided Limits

Example 2.2.5. The *Heaviside function* is the function is given by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Use a graph to determine the limit of $H(x)$ as $x \to 0$.



Limit: Does Not Exist

Other than the jump that happens at $x = 0$ (a "jump discontinuity") the Heaviside functions is fairly well-behaved for x-values near 0. We introduce the following definition:

Definition. Let f be a function and suppose that $f(x)$ is defined for all x very near the number a with $x < a$. If we can pick x-values so that $f(x)$ is arbitrarily close to some number L when x sufficiently close to a, then we write

$$\lim_{x \to a^-} f(x) = L$$

and say "the **limit** of $f(x)$, as x approaches a **from the left**, is L."

Similarly, if instead requiring that $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say "the **limit** of $f(x)$, as x approaches a **from the right**, is L."

Example 2.2.6. Let H be the Heaviside function as defined in Example 2.2.5. Find $\lim_{x \to 0^-} H(x)$ and $\lim_{x \to 0^+} H(x)$.

From the graph, it is clear that $\lim_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^+} H(x) = 1$.

The Heaviside function helps to motivate the following result.

Proposition 2.2.7. *Given a function f and a real number a,*

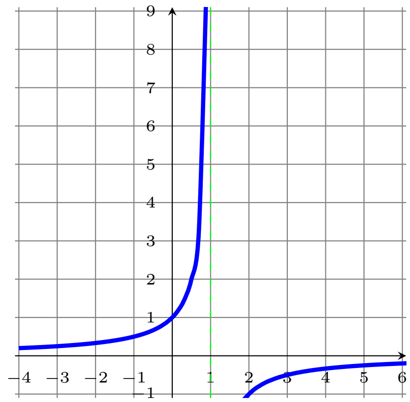
$$\lim_{x \to a} f(x) = L \qquad \text{if and only if} \qquad \text{both } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Remark. The one-sided limit notation can be used for limits involving infinity as well.

Definition. The vertical line $x = a$ is called a **vertical asymptote** for the curve $y = f(x)$ if at least one of the following is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty.$$

Example 2.2.8. Let $f(t) = \frac{1}{t-1}$. Find $\lim_{t\to 1^-} f(t)$ and $\lim_{t\to 1^+} f(t)$. List any vertical asymptotes of the graph of $f(t)$.



Notice that $f(t) < 0$ for $t < 1$, and since the denominator is getting smaller and smaller as t approaches 1 from the left, we have that

$$\lim_{t\to 1^-} f(t) = -\infty.$$

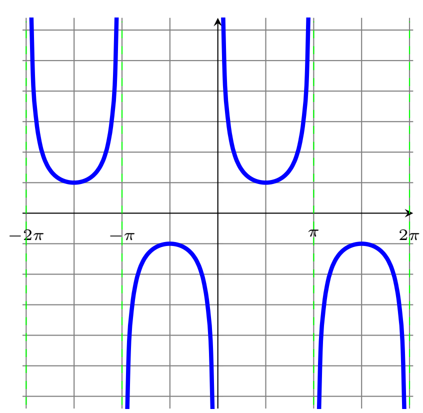
Similarly, $f(t) > 0$ for $t > 1$, and since the denominator is getting smaller and smaller as t approaches 1 from the right, we have that

$$\lim_{t\to 1^+} f(t) = \infty.$$

Indeed, $f(t)$ has only a single discontinuity at $t = 1$, so $x = 1$ is the only vertical asymptote.

Exercise 2.2.1. Let $g(t) = \dfrac{1+t}{1-t^2}$. This function is undefined both when $t = 1$ and when $t = -1$. Why is the vertical line at $t = 1$ a vertical asymptote when the vertical line at $t = -1$ is not? (It may be helpful to look at the graph and compare it to the previous example).

Example 2.2.9. Determine all vertical asymptotes of the curve $y = \csc(x)$. Recall that $\csc(x) = \dfrac{1}{\sin(x)}$. Recall also that $\sin(x) = 0$ when $x = n\pi$ for any integer n. The behavior of the left and right limits is different depending on whether n is even or odd.



Notice that $\csc(x)$ is undefined whenever $\sin(x) = 0$, i.e., whenever $x = n\pi$ for any integers n. By a similar argument as in Example 2.2.8, we see that we have vertical asymptotes at $x = n\pi$, for all integers n. This is confirmed by the graph of the function below

2.3 Calculating Limits Using the Limit Laws

Theorem 2.3.1 (Algebraic Laws of Limits)**.** *Let c be a constant and let f and g be functions such that the limits*

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. We have the following algebraic rules for limits:

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\left[\lim_{x\to a} f(x)\right]$

4. $\lim_{x\to a}[f(x)\cdot g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$

5. *If* $\lim_{x\to a} g(x) \neq 0$*, then* $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

6. *If n is a positive integer, then* $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

7. $\lim_{x\to a} c = c$

8. $\lim_{x\to a} x = a$

9. *If n is a positive integer, then* $\lim_{x\to a} x^n = a^n$

10. *If n is a positive integer, then* $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

    [If n is even, we assume $a > 0$.]

11. *If n is a positive integer, then* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

    [If n is even, we assume $\lim_{x\to a} f(x) > 0$.]

Example 2.3.2. Evaluate the following limit and justify each step: $\lim_{x\to 2}(4x^2+3)$.

$$\begin{aligned}
\lim_{x\to 2}(4x^2+3) &= \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 3 && \text{(By \# 1)}\\
&= 4\left[\lim_{x\to 2} x^2\right] + \lim_{x\to 2} 3 && \text{(By \# 3)}\\
&= 4\left[\lim_{x\to 2} x^2\right] + 3 && \text{(By \# 7)}\\
&= 4(2)^2 + 3 && \text{(By \# 9)}\\
&= 19
\end{aligned}$$

Example 2.3.3. Evaluate the following limit and justify each step: $\lim_{x\to 2} \frac{x^2+x+2}{x+1}$.

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2+x+2}{x+1} &= \frac{\lim_{x\to 2}(x^2+x+2)}{\lim_{x\to 2}(x+1)} && \text{(By \# 5)} \\
&= \frac{\lim_{x\to 2} x^2 + \lim_{x\to 2} x + \lim_{x\to 2} 2}{\lim_{x\to 2} x + \lim_{x\to 2} 1} && \text{(By \# 1)} \\
&= \frac{\lim_{x\to 2} x^2 + \lim_{x\to 2} x + 2}{\lim_{x\to 2} x + 1} && \text{(By \# 7)} \\
&= \frac{(2)^2+(2)+2)}{(2)+1} && \text{(By \# 9)} \\
&= \frac{8}{3}.
\end{aligned}$$

Proposition 2.3.4 (Direct Substitution Property)**.** *If f is a polynomial or rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.*

Remark. The trigonometric functions also satisfy this property, as do exponential functions. As we'll see, there are many general types of functions with this property.

Example 2.3.5. Find $\lim_{x\to -4} f(x)$ where $f(x) = \frac{x^2-16}{x+4}$.

Note that for limits, we only need x to be arbitrarily close to -4 and not actually equal to it. This means that $x \neq 4$, i.e., that $x+4 \neq 0$.

$$\begin{aligned}
\lim_{x\to -4} \frac{x^2-16}{x+4} &= \lim_{x\to -4} \frac{(x+4)(x-4)}{x+4} \\
&= \lim_{x\to -4} (x-4) \\
&= 8.
\end{aligned}$$

This shows us that f behaves exactly like the function $g(x) = x - 4$ everywhere except at $x = 4$.

Proposition 2.3.6. *If $f(x) = g(x)$ when $x \neq a$, then, provided the limit exists, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$*

Example 2.3.7. Find $\lim_{t \to 0} \frac{\sqrt{t+1}-1}{t}$.

Once again, with limits, we only care that t get arbitrarily close to 0. Since $t \neq 0$,

$$\begin{aligned}
\lim_{t \to 0} \frac{\sqrt{t+1}-1}{t} &= \lim_{t \to 0} \frac{\sqrt{t+1}-1}{t} \left(\frac{\sqrt{t+1}+1}{\sqrt{t+1}+1} \right) \\
&= \lim_{t \to 0} \frac{(t+1)-1}{t(\sqrt{t+1}+1)} \\
&= \lim_{t \to 0} \frac{t}{t(\sqrt{t+1}+1)} \\
&= \lim_{t \to 0} \frac{1}{\sqrt{t+1}+1} \\
&= \frac{1}{\sqrt{0+1}+1} \\
&= \frac{1}{2}.
\end{aligned}$$

Example 2.3.8. Find $\lim_{t \to 0} |t|$. First recall that

$$|t| = \begin{cases} t & \text{if } t \geq 0 \\ -t & \text{if } t < 0. \end{cases}$$

We have to use this piecewise definition of $|t|$ and take limits from the left and right, because we don't have a rule that says what to do with absolute values.

$$\begin{aligned}
\lim_{t \to 0^-} |t| &= \lim_{t \to 0^-} -t && \text{(since } t < 0\text{)} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t \to 0^+} |t| &= \lim_{t \to 0^+} t && \text{(since } t > 0\text{)} \\
&= 0
\end{aligned}$$

since $\lim_{t \to 0^-} |t| = \lim_{t \to 0^+} |t| = 0$, we must have that

$$\lim_{t \to 0} |t| = 0.$$

Lemma 2.3.9. *If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then*

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Theorem 2.3.10 (The Squeeze Theorem*). *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then*

$$\lim_{x\to a} g(x) = L.$$

Example 2.3.11. Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

First recall that the range of $\sin(t)$ is $[-1, 1]$, so $-1 \leq \sin(t) \leq 1$ for all t. So

$$\begin{aligned} -1 &\leq \sin(x) \leq 1 \\ -x^2 &\leq x^2 \sin(x) \leq x^2 \qquad \text{(since } x^2 \geq 0) \end{aligned}$$

Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, by the Squeeze theorem, we have that

$$\lim_{x\to 0} x^2 \sin(x) = 0.$$

Example 2.3.12. Find $\lim_{x\to 0} x \arctan\left(\frac{2}{x}\right)$.

Recall that the range of $\arctan(t)$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$, so $-\frac{\pi}{2} \leq \arctan(t) \leq \frac{\pi}{2}$ for all t. So, when $x \geq 0$,

$$\begin{aligned} -\frac{\pi}{2} &\leq \arctan\left(\frac{2}{x}\right) \leq \frac{\pi}{2} \\ -\frac{\pi}{2}x &\leq x \arctan\left(\frac{2}{x}\right) \leq \frac{\pi}{2}x \qquad \text{(since } x \geq 0) \end{aligned}$$

Since $\lim_{x\to O^+} -\frac{\pi}{2}x = 0$ and $\lim_{x\to 0^+} \frac{\pi}{2}x = 0$, by the Squeeze Theorem, we have that

$$\lim_{x\to 0^+} x \arctan\left(\frac{2}{x}\right) = 0.$$

Similarly, when $x < 0$,

$$\begin{aligned} -\frac{\pi}{2} &\leq \arctan\left(\frac{2}{x}\right) \leq \frac{\pi}{2} \\ -\frac{\pi}{2}x &\geq x \arctan\left(\frac{2}{x}\right) \geq \frac{\pi}{2}x \qquad \text{(since } x < 0) \end{aligned}$$

Since $\lim_{x\to O^-} -\frac{\pi}{2}x = 0$ and $\lim_{x\to 0^-} \frac{\pi}{2}x = 0$, by the Squeeze Theorem, we have that

$$\lim_{x\to 0^-} x \arctan\left(\frac{2}{x}\right) = 0.$$

Since the limits from the left and right agree,

$$\lim_{x\to 0} x \arctan\left(\frac{2}{x}\right) = 0.$$

The following result uses the Squeeze Theorem, but the proof is a bit geometric and round-about (although you can read it in Section 3.3 of the book). Instead, we'll accept it as a cool fact (but you can justify it for yourself with a table of values).

Fact. $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$

Example 2.3.13. Find $\lim\limits_{x \to 0} \dfrac{\tan(x)}{x}$.

Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$. Then

$$\begin{aligned} \lim_{x \to 0} \frac{\tan(x)}{x} &= \lim_{x \to 0} \frac{\sin(x)}{x\cos(x)} \\ &= \lim_{x \to 0} \left(\frac{\sin(x)}{x}\right)\left(\frac{1}{\cos(x)}\right) \\ &= \left(\lim_{x \to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x \to 0} \frac{1}{\cos(x)}\right) \\ &= (1)(1) \\ &= 1 \end{aligned}$$

Example 2.3.14. Find $\lim\limits_{x \to 0} \dfrac{3\sin(4x)}{5x}$.

$$\begin{aligned} \lim_{x \to 0} \frac{3\sin(4x)}{5x} &= \lim_{x \to 0} \frac{3\sin(4x)}{5x}\left(\frac{4}{4}\right) \\ &= \lim_{x \to 0} \frac{12}{5}\left(\frac{\sin(4x)}{4x}\right) \\ &= \frac{12}{5}\left(\lim_{x \to 0} \frac{\sin(4x)}{4x}\right) \end{aligned}$$

We make the substitution $t = 4x$. Then as $x \to 0$, $t \to 0$, so we get

$$\begin{aligned} &= \frac{12}{5}\left(\lim_{t \to 0} \frac{\sin(t)}{t}\right) \\ &= \frac{12}{5}(1) \\ &= \frac{12}{5}. \end{aligned}$$

*In some other countries, like Germany and Russia, the Squeeze Theorem is colloquially known as the *Two Policemen (and a Drunk) Theorem.*

2.4 The Precise Definition of a Limit

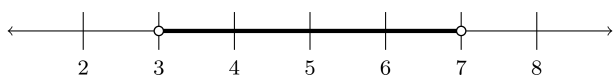
2.4.1 The meaning of $|x - a| < \delta$ and $|y - L| < \varepsilon$

In what follows, we will be using the greek letters δ (delta) and ε (epsilon) quite a bit. Much like x, y, these will just be names given to real variables and they will not have any inherent value like $\pi = 3.141592\ldots$. This notation is rather traditional and is used in many advanced courses, so we will honor our ancestors and keep with the tradition.

If I asked you shade $|x - 5| < 2$ on a number line, you would probably do the following

$$|x - 5| < 2 \iff -2 < x - 5 < 2 \iff 5 - 2 < x < 5 + 2 \iff 3 < x < 7.$$

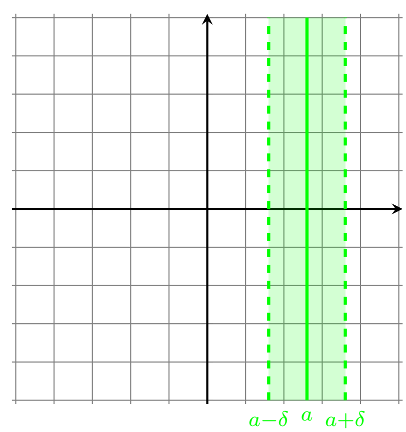
and then shade the number line



The thing to notice is that this is the set of all values that are *at most* 2 units away from 5. The same idea works in the Cartesian plane where we consider all (x, y) pairs where the x-coordinate is at most 2 units away from 5 (so the picture is a vertical strip centered around the line $x = 2$). The pictures to have in mind are the following:

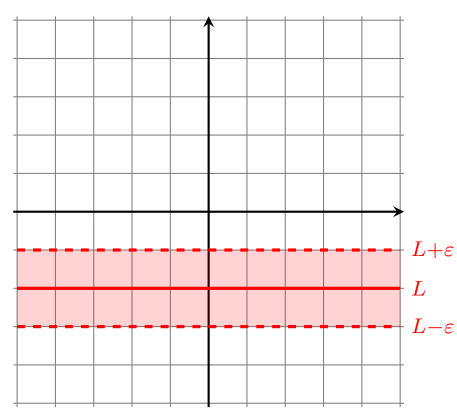
For some real number a and some positive real number $\delta > 0$

$$|x - a| < \delta$$



Similarly, for some real number L and some positive real number $\varepsilon > 0$

$$|y - L| < \varepsilon$$



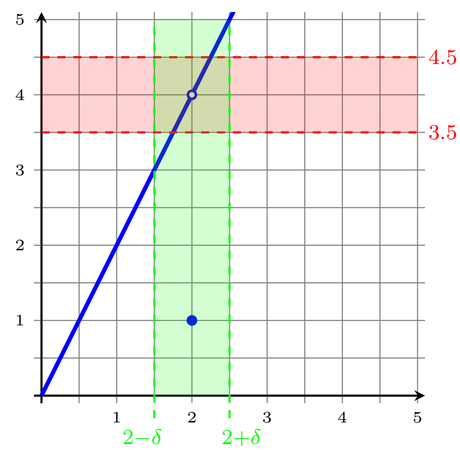
2.4.2 Understanding the Definition Graphically

Conceptually, limits tell us the behavior of a function near a point (even if that function is not actually defined at that point), and we calculate limits by "zooming in" and approximating them with better and better precision. So how do we make this rigorous? We can motivate it with the following example.

Example 2.4.1. Let

$$f(x) = \begin{cases} 2x & \text{when } x \neq 2, \\ 1 & \text{when } x = 2 \end{cases}$$

We know that when x is close to 2 (but $x \neq 2$), then $f(x)$ is close to 4. Find a range of x-values can we use to approximate $f(x)$ within 0.5 of 4.



Since $|x-2|$ is the distance from x to 2 and $|f(x)-4|$ is the distance from $f(x)$ to 2, we can rephrase the problem as

Find a positive real number δ so that $|f(x)-2| < 0.5$ whenever $0 < |x-2| < \delta$.

We notice from the graph that if $|x-2| < 0.25$, then $|f(x)-4| < 0.5$. Indeed we can verify this explicitly:

$$|f(x)-4| = |2x-4| = |2(x-2)| = 2|x-2| < 2 * 0.25 = 0.5$$

and so $\delta = 0.25$ is perfectly reasonable. In fact, for any choice of δ less than 0.25 is valid as well:

$$|f(x)-4| = |2x-4| = |2(x-2)| = 2|x-2| < 2 * \delta < 2 * 0.25 = 0.5,$$

so any choice of δ in the range $0 < \delta < 0.25$ will approximate $f(x)$ within 0.5 of the value 2.

Remark. At the time of this writing, one can access an interactive graph of the above example at https://www.desmos.com/calculator/qqspjadxfd.

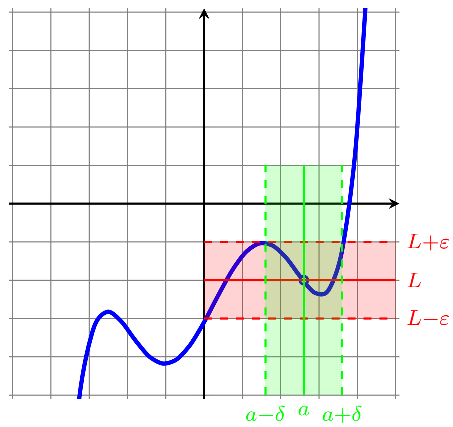
Definition. Let f be a function defined on an open interval containing a, except possibly at a itself, and let L be a real number. Then we say that **the limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L,$$

if for every positive real number ε (epsilon) there exists some $\delta > 0$ such that

if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Pictorially, this definition says that, given any horizontal ε-neighborhood centered at the line $y = L$, we can find some vertical δ-neighborhood centered at $x = a$ so that the graph $y = f(x)$ contained within the horizontal window *sits entirely inside the vertical window as well.*

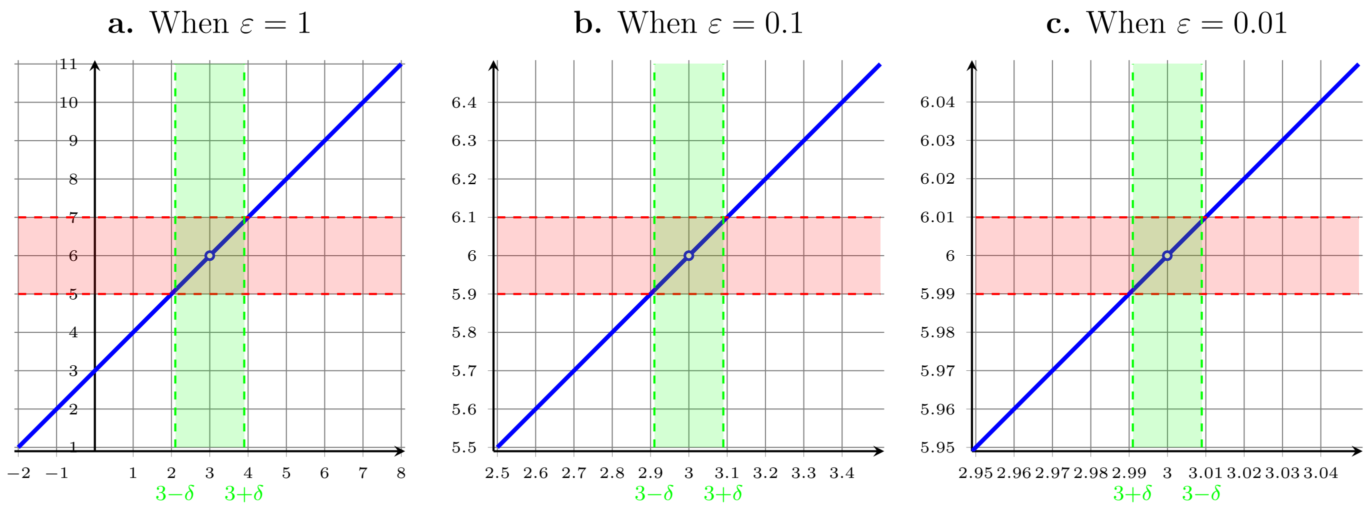


Remark. It's hopefully clear from the picture, but if you can find a suitable $\delta > 0$, then the result will also hold for any smaller positive value $0 < \tilde{\delta} < \delta$. In other words, *the choice of delta is not unique*, but when possible, we will try to find the largest value of δ that works.

Remark. It took more than 100 years after the invention of calculus to make rigorous the definition of a limit (although exact dates and author attributions are debated).

Example 2.4.2. Let $f(x) = \dfrac{x^2 - 9}{x - 3}$. Graph f and use this to find a value of δ so that

- **a.** if $0 < |x - 3| < \delta$ then $|f(x) - 6| < 1$.
- **b.** if $0 < |x - 3| < \delta$ then $|f(x) - 6| < 0.1$.
- **c.** if $0 < |x - 3| < \delta$ then $|f(x) - 6| < 0.01$.
- **d.** if $0 < |x - 3| < \delta$ $|f(x) - 6| < \varepsilon$ (this means that δ should be related to ε in some way).



a. From the graph above, it seems that picking any value of $\delta \le 1$ will work. We justify this analytically. Whenever $0 < |x-3|$, then $x \neq 3$ and thus $f(x) = x+3$. It follows that $|f(x)-6| = |x-3|$. Choosing $\delta \le 1$, we have

$$|f(x)-6| = |x-3| < \delta \le 1.$$

b. From the graph above, it seems that picking any value of $\delta \le 0.1$ will work. We justify this analytically. Whenever $0 < |x-3|$, then $x \neq 3$ and thus $f(x) = x+3$. It follows that $|f(x)-6| = |x-3|$. Choosing $\delta \le 0.1$, we have

$$|f(x)-6| = |x-3| < \delta \le 0.1.$$

c. From the graph above, it seems that picking any value of $\delta \le 0.01$ will work. We justify this analytically. Whenever $0 < |x-3|$, then $x \neq 3$ and thus $f(x) = x+3$. It follows that $|f(x)-6| = |x-3|$. Choosing $\delta \le 0.01$, we have

$$|f(x)-6| = |x-3| < \delta \le 0.01.$$

d. From the previous parts, we would guess that picking any value of $\delta \le \varepsilon$ will work. We justify this analytically. Whenever $0 < |x-3|$, then $x \neq 3$ and thus $f(x) = x+3$. It follows that $|f(x)-6| = |x-3|$. Choosing $\delta \le \varepsilon$, we have

$$|f(x)-6| = |x-3| < \delta \le \varepsilon.$$

You won't be expected to prove things rigorously in this course, but you may be happy to know that you basically did just prove the next example.

Example 2.4.3. Prove that $\lim_{x\to 3} \dfrac{x^2-9}{x-3} = 6$.

Proof. Let $\varepsilon > 0$, and choose $\delta = \varepsilon$. Suppose that $0 < |x-3| < \delta$. Then

$$|f(x)-6| = \left|\frac{x^2-9}{x-3} - 6\right| = \frac{|x-3|^2}{|x-3|} = |x-3| < \delta = \varepsilon.$$

Therefore $\lim_{x\to 3} \dfrac{x^2-9}{x-3} = 6$. □

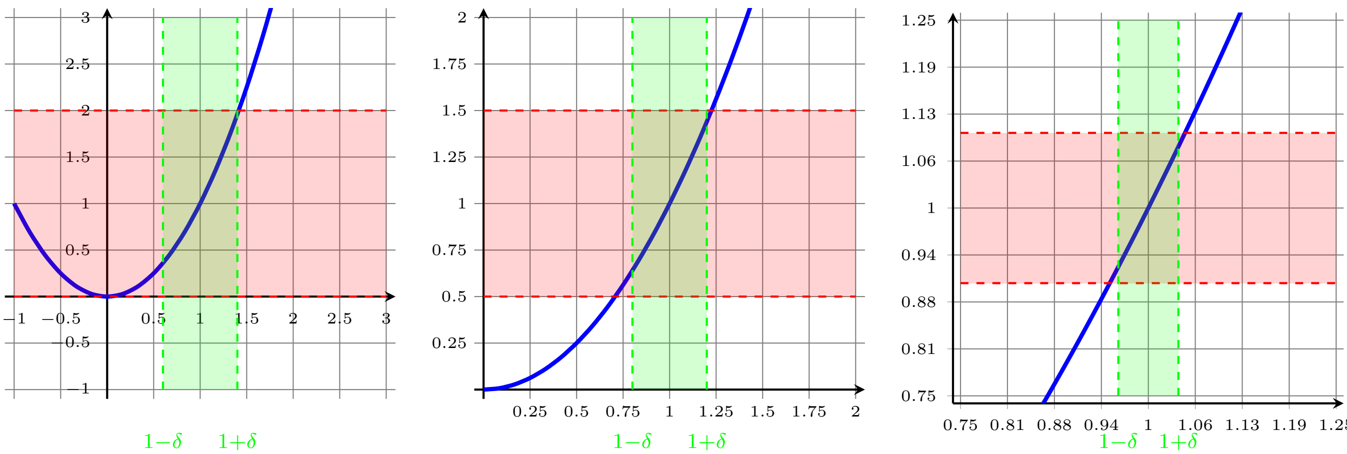
Example 2.4.4. Let $f(x) = x^2$. Find Find a value of δ so that

a. $|f(x) - 1| < 1$ whenever $0 < |x - 1| < \delta$.

b. $|f(x) - 1| < 0.5$ whenever $0 < |x - 1| < \delta$.

c. $|f(x) - 1| < 0.1$ whenever $0 < |x - 1| < \delta$.

a. When $\varepsilon = 1$ **a.** When $\varepsilon = 0.5$ **c.** When $\varepsilon = 0.1$



a. From the graph above, it seems that picking any value of $\delta \leq 0.41$ will work. We justify this analytically. Note that in this δ-window, $x < 1.41$ and thus $|x+1| < 2.41$. Choosing $\delta \leq 0.41$, we have that

$$|f(x) - 1| = |x^2 - 1| = |x+1||x-1| < 2.41 \cdot \delta \leq 2.41 \cdot 0.41 = 0.9881 < 1$$

b. From the graph above, it seems that picking any value of $\delta \leq 0.22$ will work. We justify this analytically. Note that in this δ-window, $x < 1.22$ and thus $|x+1| < 2.22$. Choosing $\delta \leq 0.22$, we have that

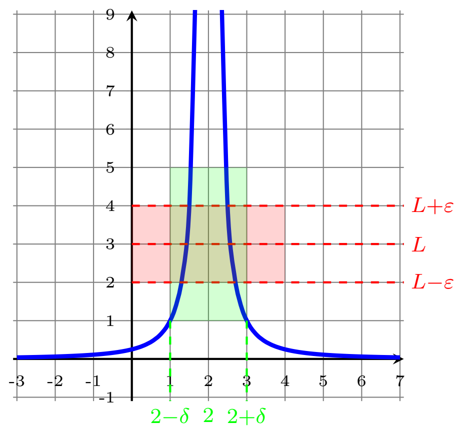
$$|f(x) - 1| = |x^2 - 1| = |x+1||x-1| < 2.22 \cdot \delta \leq 2.22 \cdot 0.22 = 0.4884 < 0.5$$

c. From the graph above, it seems that picking any value of $\delta \leq 0.04$ will work. We justify this analytically. Note that in this δ-window, $x < 1.04$ and thus $|x+1| < 2.04$. Choosing $\delta \leq 0.04$, we have that

$$|f(x) - 1| = |x^2 - 1| = |x+1||x-1| < 2.04 \cdot \delta \leq 2.04 \cdot 0.04 = 0.0816 < 0.1$$

Remark. If you're bored and want to try writing a rigorous proof of Example 2.4.4 above, it suffices to restrict your attention to $\varepsilon \leq 1$, and then the optimal choice is $\delta = \sqrt{1+\varepsilon} - 1$.

Example 2.4.5. Appeal to the precise definition of a limit to explain why the function $f(x) = \dfrac{1}{(x-2)^2}$ does not have a limit as x approaches 2.

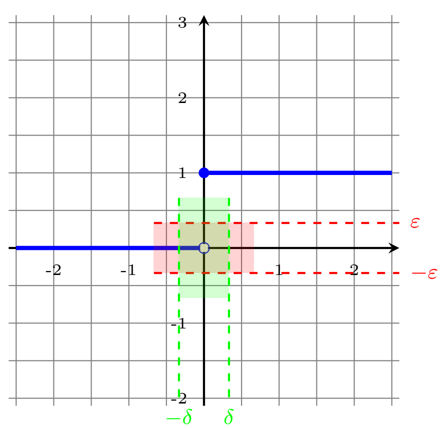


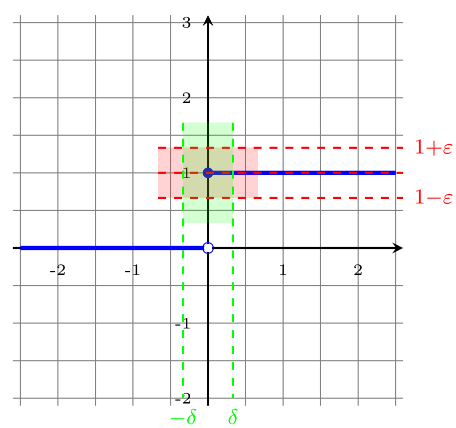
From the graph we see that, no matter what we conjecture the limit will be, for any given horizontal ε-window, no matter what δ we choose, the portion of the function in the vertical δ-window escapes the horizontal ε-window.

Example 2.4.6 (Heaviside function)**.** Appeal to the precise definition of a limit to explain why the function

$$f(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}$$

does not have a limit as x approaches 0.





It's reasonable to guess that, if $H(x)$ had a limit as x approaches 0, the limit would be either 0 or 1. However, notice that for any sufficiently small value of ε (say $\varepsilon < \frac{1}{3}$), no matter what choice of δ we make for our vertical δ-window, there some part of the function that is missing from the horizontal ε-window. In fact, this is true for whatever value we try to assign to the limit: For any real number L and any $\delta > 0$, there is always a nonzero value of x satisfying $0 < |x| < \delta$ and $|H(x) - L| \geq \varepsilon$.

2.4.3 Infinite Limits

Infinite limits are defined similarly, except instead of the horizontal ε-window, we appeal to the notion of "growing without bound."

Definition. Let f be a function defined on an open interval containing a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty,$$

if for every positive real number M there exists some $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \text{ then } f(x) > M.$$

Definition. Let f be a function defined on an open interval containing a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = -\infty,$$

if for every negative real number N there exists some $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \text{ then } f(x) < N.$$

Example 2.4.7. Let $f(x) = \dfrac{1}{(x-2)^2}$. Prove that $\lim_{x\to 2} f(x) = \infty$.

We want to find δ so that $f(x) > M$ whenever $|x-2| < \delta$. Notice that

$$\frac{1}{(x-2)^2} > M \quad \Rightarrow \quad (x-2)^2 < \frac{1}{M} \quad \Rightarrow \quad |x-2| < \frac{1}{\sqrt{M}}.$$

We guess that $\delta = \frac{1}{\sqrt{M}}$ is probably what we want.

Proof. Let $M > 0$ and choose $\delta = \frac{1}{\sqrt{M}}$. Suppose that $|x-2| < \delta$. Then

$$f(x) = \frac{1}{(x-2)^2} > \frac{1}{\delta^2} = \frac{1}{1/M} = M.$$

Thus $f(x)$ grows without bound as x approaches 2, and therefore $\lim_{x\to 2} f(x) = \infty$. □

2.5 Continuity

Definition. A function f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$.

Remark. The definition above implicitly requires the following three things if f is continuous at a:

- $\lim_{x\to a} f(x)$ exists
- $f(a)$ exists
- The two values agree.

Intuitively, it means that we can draw the graph of f (near a) without having to lift the pencil off of the page.

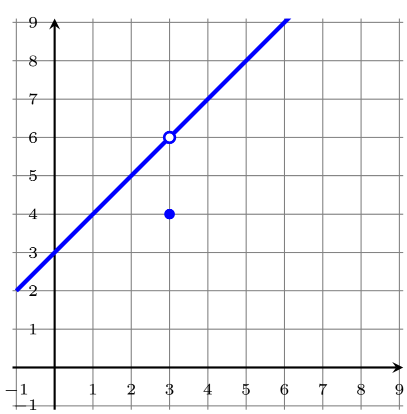
Definition. If $f(x)$ is defined for all x in an open interval containing a, but f is not continuous at a, we say that f is **discontinuous at** a, or alternatively that f has a **discontinuity at** a.

Remark. The word "discontinuous" will be used *only for points that are in the function's domain.* If an x-value is not in the function's domain, we will just say that the function is *not continuous* at that point. This is a subtlety, but the distinction will make a difference later.

All discontinuities are not created equal.

Example 2.5.1. Where is the following function discontinuous? Graph the function.

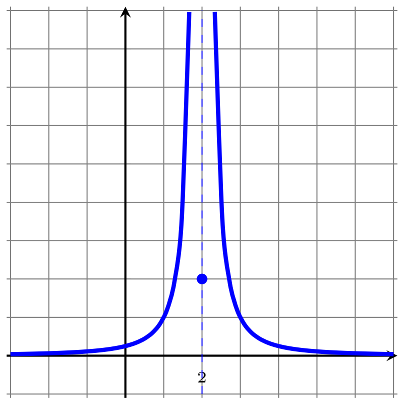
$$f(t) = \begin{cases} \dfrac{t^2-9}{t-3} & \text{when } t \neq 3, \\ 4 & \text{when } t = 3. \end{cases}$$



Discontinuity: $t = 3$

Example 2.5.2. Where is the following function discontinuous? Graph the function.

$$f(x) = \begin{cases} \dfrac{1}{(x-2)^2} & \text{when } x \neq 2, \\ 2 & \text{when } x = 2. \end{cases}$$

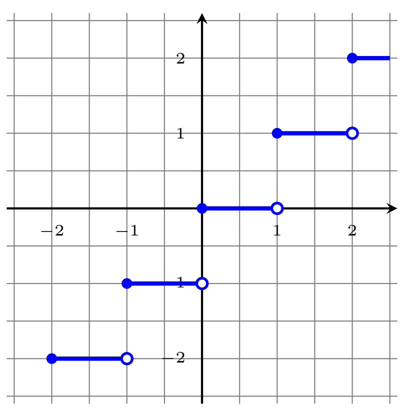


Discontinuity: $x = 2$

Example 2.5.3. Where is the following function discontinuous? Graph the function.

$$f(x) = \lfloor x \rfloor$$

This function is called the "floor function". It rounds a number down to the nearest integer.



Discontinuities:
$x = n$, for every integer n

Definition. Let f be a function that is discontinuous at a.

- We say that a is a **removable discontinuity** if we can find a function g such that g is continuous at a and $f(x) = g(x)$ for all $x \neq a$. [See Example 2.5.1]
- We say that a is an **infinite discontinuity** if $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ are unbounded (i.e. "go to $\pm\infty$"). [See Example 2.5.2]
- We say that a is a **jump discontinuity** if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ when both one-sided limits exist. [See Example 2.5.3]

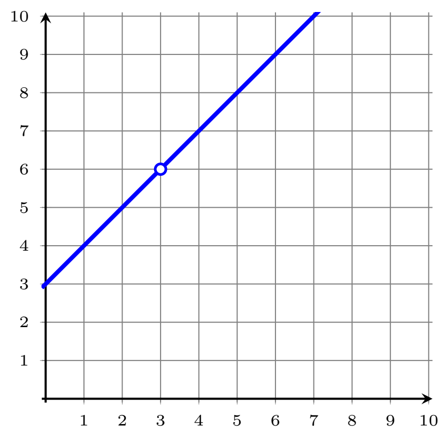
Definition. Suppose f is a function that is not continuous at a. If g is some other function that is continuous at a and $f(x) = g(x)$ (for all $x \neq a$), then we say that g is a **continuous extension** of f.

Example 2.5.4. Let $f(x) = \dfrac{x^2 - 9}{x - 3}$. Find a continuous extension for f.

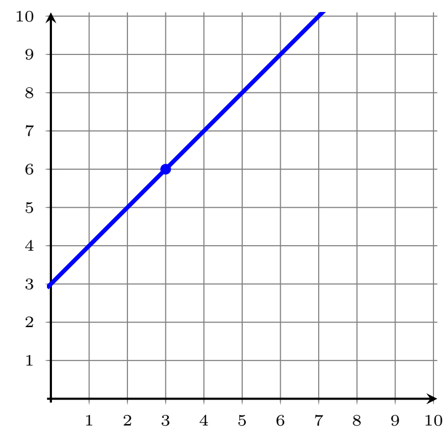
We see that f is not continuous at $x = 3$. Moreover, when $x \neq 3$,

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = x + 3.$$

The function $g(x) = x + 3$ is continuous at 3 and $f(x) = g(x)$ for all $x \neq 3$, so g is a continuous extension for f.



Graph of f.



Graph of g.

Definition. A function f is **continuous from the right at** a if

$$\lim_{x \to a^+} f(x) = f(a),$$

and is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example 2.5.5. The floor function from Example 2.5.3 is continuous from the right, but not from the left.

To see this, let $a = n$ for any integer n. We then have that

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \neq \lfloor n \rfloor,$$

so at each n, $\lfloor x \rfloor$ is not continuous from the left. However,

$$\lim_{x \to n^+} \lfloor x \rfloor = n = \lfloor n \rfloor,$$

so at each n, $\lfloor x \rfloor$ is continuous from the right.

Definition. A function f is **continuous on the interval...**

- $\boldsymbol{(a, b)}$ if it is continuous at every number in that interval.
- $\boldsymbol{[a, b)}$ if it is continuous on (a, b) and continuous from the right at a.
- $\boldsymbol{(a, b]}$ if it is continuous on (a, b) and continuous from the left at b.
- $\boldsymbol{[a, b]}$ if it is continuous on both $[a, b)$ and $(a, b]$.

Remark. A function f is simply said to be *continuous* if it is continuous on its whole domain.

Theorem 2.5.6 (Algebraic Laws of Continuous Functions)**.** *Let c be some constant and let f and g be functions. Suppose that f and g are continuous at a. Then each of the following functions are continuous at a:*

1. $f + g$, *where* $(f + g)(x) = f(x) + g(x)$
2. $f - g$, *where* $(f - g)(x) = f(x) - g(x)$
3. cf, *where* $(cf)(x) = cf(x)$
4. fg, *where* $(fg)(x) = f(x)g(x)$
5. $\dfrac{f}{g}$ *if* $g(a) \neq 0$, *where* $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

We will prove that (fg) is continuous at a, but the rest all follow similarly and are left as an exercise to the reader.

Proof of 4. Since f and g are both continuous at a, we have that

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a).$$

Since these limits exist, we can apply our algebraic laws of limits to get

$$\begin{aligned}\lim_{x \to a}(fg)(x) &= \lim_{x \to a}\left[f(x)g(x)\right] \\ &= \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] \\ &= f(a)g(a) = (fg)(a),\end{aligned}$$

and so fg is continuous at a. □

Proposition 2.5.7. *Every polynomial $p(x) = c_n x^n + \cdots + c_1 x + c_0$ is continuous on $(-\infty, \infty)$.*

Proof. Certainly

$$p(a) = c_n a^n + \cdots + c_1 a + c_0$$

is defined for every real number a. So, using our algebraic limit laws, we get

$$\begin{aligned}\lim_{x \to a} p(x) &= \lim_{x \to a}\left(c_n x^n + \cdots + c_1 x + c_0\right) \\ &= \lim_{x \to a}\left(c_n x^n\right) + \cdots + \lim_{x \to a}\left(c_1 x\right) + \lim_{x \to a} c_0 \\ &= c_n\left(\lim_{x \to a} x^n\right) + \cdots + c_1\left(\lim_{x \to a} x\right) + \lim_{x \to a} c_0 \\ &= c_n a^n + \cdots + c_1 a + c_0 = p(a),\end{aligned}$$

and therefore $p(x)$ is continuous at every real number a. □

Corollary 2.5.8. *Every rational function* $f(x) = \dfrac{p(x)}{q(x)}$*, where* p *and* q *are polynomials, is continuous on its domain.*

Fact. Root functions, trigonometric functions, exponential functions, and logarithmic functions are all continuous on their domains.

The following theorem demonstrates an important fact about the interplay between limits and continuous functions.

Theorem 2.5.9. *If* f *is continuous at* b *and* $\lim_{x\to a} g(x) = b$*, then* $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$

Proof. See Appendix F. ☐

Proposition 2.5.10. *If* g *is continuous at* a *and* f *is continuous at* $g(a)$*, then the composite function* $f \circ g$ *given by* $(f \circ g)(x) = f(g(x))$ *is continuous at* a*.*

Proof. Since g is continuous at a,

$$\lim_{x\to a} g(x) = g(a).$$

Since f is continuous at $g(a)$, then by Theorem 2.5.9

$$\begin{aligned}\lim_{x\to a}(f \circ g)(x) &= \lim_{x\to a} f(g(x)) \\ &= f\left(\lim_{x\to a} g(x)\right) \\ &= f(g(a)) \\ &= (f \circ g)(a),\end{aligned}$$

and therefore $f \circ g$ is continuous at a. ☐

Example 2.5.11. On what interval(s) is $f(x) = \sqrt{1 + \frac{1}{x}}$ continuous? Why?

We know that 1, $\frac{1}{x}$, and $\sqrt{x}$ are all continuous on their domains. By the Algebraic Laws of Continuous Functions (Theorem 2.5.6) and Proposition 2.5.10, it follows that $f(x) = \sqrt{1 + \frac{1}{x}}$ is continuous on its domain. The domain of f is $(-\infty, -1] \cup (1, \infty)$, and so f is continuous on these intervals.

Example 2.5.12. Show that $f(x) = \begin{cases} 1 - x^2 & \text{when } x \le 1 \\ \ln(x) & \text{when } x > 1 \end{cases}$ is continuous on $(-\infty, \infty)$.

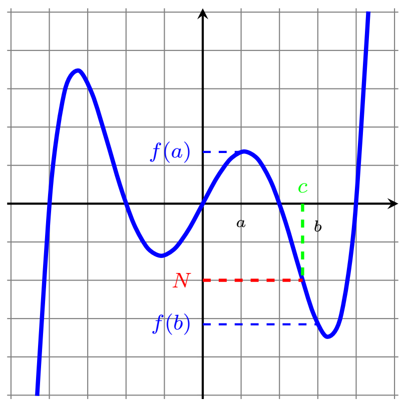
Because $1 - x^2$ and $\ln(x)$ are both continuous on their domains, we have that f is continuous on $(-\infty, 1]$ and on $(1, \infty)$ separately. It suffices to show that f is continuous at 1.

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 1 - x^2 = 0 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \ln(x) = 0 \\ f(1) &= 0\end{aligned}$$

therefore f is continuous at 1 also.

Theorem 2.5.13 (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in the open interval (a, b) such that $f(c) = N$.*

Although the proof of this theorem is beyond the scope of the course, we can demonstrate with a graph below why it intuitively makes sense.



Another way to think about it is to ask the following question (incredulously). "If f is continuous, how can we possibly draw its graph *without* crossing the line $y = N$?"

Example 2.5.14. Let $g(t) = \log_2(t)$. Use the Intermediate Value Theorem to show that there is some c in the interval $(1, 64)$ so that $g(c) = 5$. Can you find such a c explicitly?

Since $\log_2(t)$ is continuous on $(0, \infty)$, it is certainly continuous on $[1, 64]$. We have that $f(1) = 0 < 5 < 6 = f(64)$, and so by IVT there must be some c in the interval $(1, 64)$ where $g(c) = 5$.

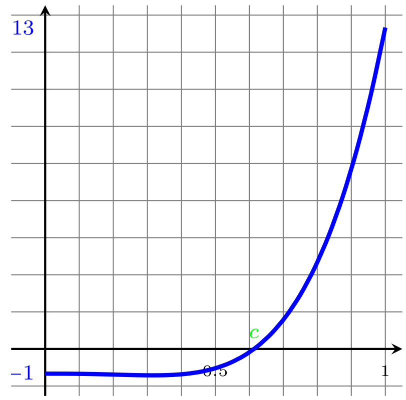
Explicitly, we know that $32 = 2^5$, and so $g(32) = \log_2(32) = \log_2(2^5) = 5$.

Example 2.5.15. Let $f(x) = \dfrac{x^2 + 2x - 3}{x^2 + 1}$. Use the Intermediate Value Theorem to show that there is some c in the interval $(0, 4)$ for which $f(c) = 0$. Can you find such an c explicitly?

f is a rational function and is continuous on its domain, which is $(-\infty, \infty)$, so it must also be continuous on $[0, 4]$. Since $f(0) = -3 < 0 < \frac{21}{17} = f(4)$, then by the Intermediate Value Theorem, there must be some c in the interval $(0, 4)$ for which $f(c) = 0$.

Explicitly, we can factor the numerator as $x^2 + 2x - 3 = (x + 3)(x - 1)$, and from this we see that $f(1) = 0$.

Example 2.5.16. Show that the polynomial function $p(x) = 5x^5 + 17x^4 - 8x^3 - 1$ has a zero between 0 and 1.



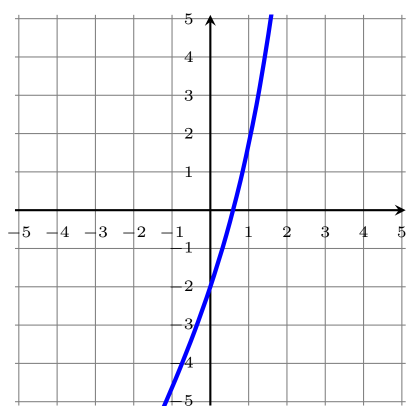
Since $p(x)$ is continuous on its domain $(-\infty, \infty)$, it is certainly continuous on $[0, 1]$. Since

$$p(0) = -1 < 0 < 13 = p(1),$$

by the Intermediate Value Theorem there must exist some real number $0 < c < 1$ such that $p(c) = 0$.

Note: We can't solve for it explicitly, but from the graph, we can see $c \approx 0.6$.

Example 2.5.17. Let $h(x) = e^x + 2x - 3$. Show that there is some c where $h(c) = 0$.



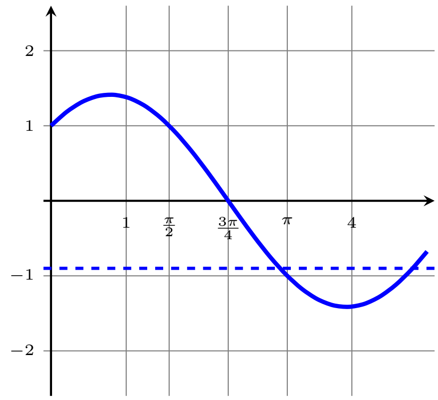
Notice we weren't given an interval this time, so we must find one. It's not too hard to see that $h(0) = -3$ and $h(1.5) = e^{1.5} > 0$, so we might try checking the interval $[0, 1.5]$. Indeed, $h(x)$ is continuous on its domain (it is a combination of continuous functions as per the Algebraic Laws of Continuous Functions 2.5.6), so it is certainly continuous on $[0, 1.5]$. Since

$$h(0) = -3 < 0 < e^{1.5} = h(1.5)$$

by the Intermediate Value Theorem there must exist some real number $0 < c < 1.5$ where $h(c) = 0$.

Note: We can't solve for c explicitly, but from a graph of the function we can estimate that $c \approx 0.6$.

Example 2.5.18. Let $f(x) = \cos(x) + \sin(x)$. Show that there is some number c in the interval $(1, 4)$ where $f(x) = -0.9$.



We don't know the value of $f(1)$ or $f(4)$, but there are some values of x in this interval that we do know, namely, $1 < \frac{\pi}{2} < \frac{3\pi}{4} < \pi < 4$. A quick computation shows that $f\left(\frac{\pi}{2}\right) = 1$ and $f(\pi) = -1$, so it may be reasonable to consider the interval $\left[\frac{\pi}{2}, \pi\right]$. As a sum of continuous function, f is continuous on all real numbers, hence is continuous on the interval $\left[\frac{\pi}{2}, \pi\right]$. Since

$$f\left(\frac{\pi}{2}\right) = 1 > -0.9 > -1 = f(\pi)$$

then by the Intermediate Value Theorem, there is some c in the interval $\left(\frac{\pi}{2}, \pi\right)$ [and thus also in the interval $(1, 4)$] where $f(c) = -0.9$.

Note: We can't solve for c explicitly, but from the graph we see that $c \approx 3$.

Example 2.5.19. Let $g(x) = \frac{1}{x} + x^2$. Show that there is some number c where $g(c) = -\dfrac{1}{100}$.

Once again, we should try to find an interval. We know that $g(x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$, so we should restrict our attention to just one of these intervals. Also, because we're wanting $g(x)$ to be negative, we should restrict our attention to $(-\infty, 0)$. Without too much work, we see that $g(-1) = 0$ and $g\left(-\frac{1}{10}\right) = -\frac{999}{100} < -\frac{1}{100}$, so we might consider restricting our attention to the interval $\left[-1, -\frac{1}{10}\right]$. Certainly g is continuous on this interval, so since

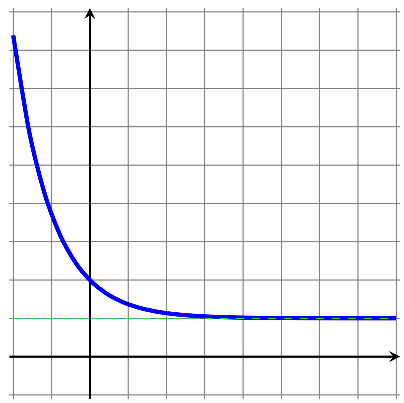
$$g(-1) = 0 > -\frac{1}{100} > -\frac{999}{100} = g\left(-\frac{1}{10}\right)$$

by the Intermediate Value Theorem, there is some c in the interval $\left(-1, -\frac{1}{10}\right)$ where $g(c) = 0$.

Note: We can't solve for c explicitly, but from the graph we see that $c \approx -0.99$.

2.6 Limits at Infinity; Horizontal Asymptotes

Example 2.6.1. Graph the function $f(x) = e^{-x} + 1$. What do you notice about about $f(x)$ as x gets arbitrarily large?



As x becomes larger and larger, $f(x)$ gets closer and closer to 1. Indeed, we can get arbitrarily close to 1 by choosing sufficiently large x. So, the following notation seems reasonable:

$$\lim_{x\to\infty} f(x) = 1.$$

Indeed, we present the following definitions.

Definition. Suppose f is defined on the interval (a, ∞). If L is a real number such that $f(x)$ is sufficiently close to L as x becomes arbitrarily larger than a, then we write

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, suppose f is defined on the interval $(-\infty, a)$. If L is a real number such that $f(x)$ is sufficiently close to L as x becomes arbitrarily smaller than a, then we write

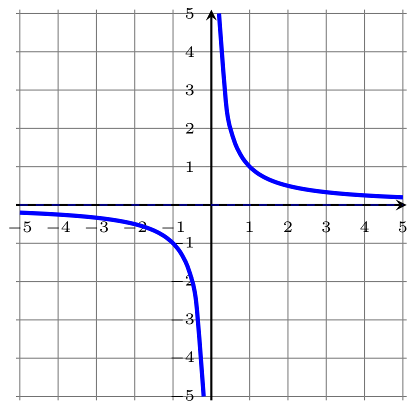
$$\lim_{x\to-\infty} f(x) = L.$$

Remark. Because all real numbers are less than infinity, we can only ever really "approach ∞" from the left. Similarly, we can only ever really "approach $-\infty$" from the right. For this reason, we typically do not use the "from the left" or "from the right" limit notation when looking at the limits above.

Definition. The horizontal line $y = L$ is a **horizontal asymptote** for the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Example 2.6.2. Find $\lim_{x\to\infty} \dfrac{1}{x}$ and $\lim_{x\to-\infty} \dfrac{1}{x}$.



Notice that as x tends toward ∞, the fraction $\frac{1}{x}$ stays positive but becomes smaller and smaller, hence

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Similarly, as x tends toward $-\infty$, fraction $\frac{1}{x}$ stays negative but becomes smaller and smaller, hence

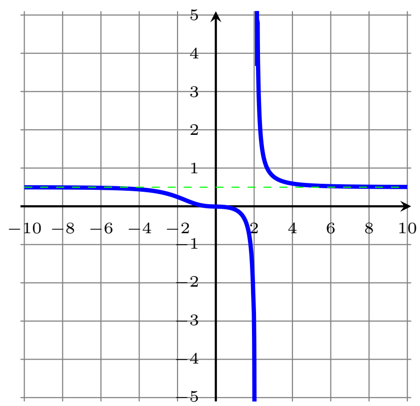
$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

Proposition 2.6.3. *For any rational number* $r > 0$

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad \textit{and} \quad \lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Proof. The proof follows from the previous example and the product/roots of limits property. □

Example 2.6.4. Find any horizontal asymptotes for the function $f(x) = \dfrac{8x^3 + 2x + 1}{16x^3 - 147}$.



To find the horizontal asymptotes, we must take limits as $x \to \pm\infty$.

$$\begin{aligned}\lim_{x\to-\infty} \frac{8x^3+2x+1}{16x^3-147} &= \lim_{x\to-\infty} \frac{x^3\left(8+\frac{2}{x^2}+\frac{1}{x^3}\right)}{x^3\left(16-\frac{147}{x^3}\right)} \\ &= \lim_{x\to-\infty} \frac{8+\frac{2}{x^2}+\frac{1}{x^3}}{16-\frac{147}{x^3}} \\ &= \frac{8+0+0}{16-0} \\ &= \frac{8}{16} = \frac{1}{2}.\end{aligned}$$

By a similar argument, we have $\lim_{x\to\infty} f(x) = \frac{1}{2}$, hence we have a single horizontal asymptote $y = \frac{1}{2}$.

The technique applied in this example proves the following proposition.

Proposition 2.6.5. *Consider the rational function*

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0},$$

where m, n are positive integers, $a_0, \ldots, a_n, b_0, \ldots, b_m$ are real numbers, and a_n, b_m are nonzero.

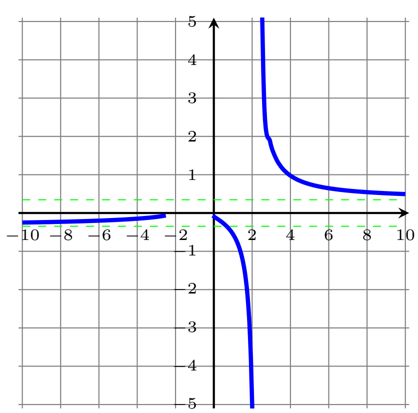
- *If $n > m$, then f has no horizontal asymptotes.*
- *If $n = m$, then f has one horizontal asymptote: $y = \frac{a_n}{b_m}$.*
- *If $n < m$, then f has one horizontal asymptote: $y = 0$.*

Before this next example, we'll recall the following fact

Fact. For any real number x,

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{when } x \geq 0, \\ -x & \text{when } x < 0. \end{cases}$$

Example 2.6.6. Find all horizontal asymptotes of the function $f(x) = \dfrac{\sqrt{3x^2+7x+1}}{5x-11}$.



We'll begin by finding the limit as $x \to -\infty$.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{3x^2+7x+1}}{5x-11} &= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(3+\frac{7}{x}+\frac{1}{x^2}\right)}}{x\left(5-\frac{11}{x}\right)} \\
&= \lim_{x\to-\infty} \frac{|x|\sqrt{3+\frac{7}{x}+\frac{1}{x^2}}}{x\left(5-\frac{11}{x}\right)} \\
&= \lim_{x\to-\infty} -\frac{\sqrt{3+\frac{7}{x}+\frac{1}{x^2}}}{5-\frac{11}{x}} \quad (\text{since } |x| = -x \text{ for } x<0) \\
&= \lim_{x\to-\infty} -\frac{\sqrt{3+0+0}}{5-0} \\
&= -\frac{\sqrt{3}}{5},
\end{aligned}$$

and a similar argument shows us that $\lim_{x\to\infty} \frac{\sqrt{3x^2+7x+1}}{5x-11} = \frac{\sqrt{3}}{5}$. So we have two horizontal asymptotes: $x = -\frac{\sqrt{3}}{5}$ and $x = \frac{\sqrt{3}}{5}$.

Example 2.6.7. Find $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$.

By Proposition 2.6.3, we have that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

and so by Theorem 2.5.9, we have that

$$\begin{aligned}
\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) &= \cos\left(\lim_{x\to\infty} \frac{1}{x}\right) \\
&= \cos(0) \\
&= 1.
\end{aligned}$$

Example 2.6.8. Find $\lim_{x\to\infty} \cos(x)$ if it exists.

Notice that as x tends toward ∞, $\cos(x)$ achieves every value in the interval $[-1, 1]$ infinitely many times. As such, $\cos(x)$ never tends toward any single real number and the limit does not exist.

2.6.1 Infinite Limits at Infinity

Definition. Given a function f, the following notation indicates that the value $f(x)$ grows without bound (positively or negatively) as $x \to \infty$ or $x \to -\infty$:

$$\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to-\infty} f(x) = \infty$$

$$\lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

Again, we're *not* saying that ∞ is a number or that $f(x)$ has a horizontal asymptote $y = \infty$. This notation is merely suggestive of the behavior of the graph.

Remark. Although we're throwing around ∞ all over the place, we have to remember to exercise caution; we cannot treat it like a real number and/or blindly apply our algebraic limit laws as though it were.

Example 2.6.9 ("∞/∞" indeterminate form)**.** Consider $f(x) = \frac{x+2}{3x+4}$, $g(x) = \frac{5x^2+6}{7x+8}$, and $h = \frac{9x+10}{11x^2+12}$. We may notice that the limits of the numerators and the limits of the denominators of each of these functions is ∞ as $x \to \infty$. For that reason, we carelessly using the above notation, we might be tempted to write

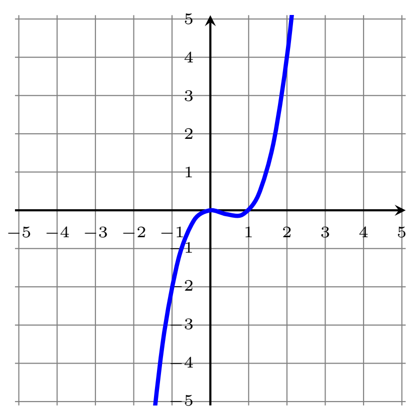
$$\lim_{x\to\infty} f(x) = \frac{\infty}{\infty} \qquad \lim_{x\to\infty} g(x) = \frac{\infty}{\infty} \qquad \lim_{x\to\infty} h(x) = \frac{\infty}{\infty}$$

and then we would probably say that $\frac{\infty}{\infty} = 1$. However, by Proposition 2.6.5, we know that the limits actually are

$$\lim_{x\to\infty} f(x) = \frac{1}{3} \qquad \lim_{x\to\infty} g(x) = 0 \qquad \lim_{x\to\infty} h(x) = \text{DNE}.$$

Since $\frac{\infty}{\infty}$ cannot be equal to all of these things, it must be undefined.

Example 2.6.10 ("$\infty - \infty$" indeterminate form)**.** Consider $f(x) = x^3 - x^2$, and suppose we want to find $\lim_{x\to\infty} f(x)$.



If we could apply the algebraic laws of limits, we would have

$$\lim_{x\to\infty} x^3 - x^2 = \lim_{x\to\infty} x^3 - \lim_{x\to\infty} x^2 = \infty - \infty.$$

But "$\infty - \infty$" is undefined. You may be tempted to make it 0, but then this would disagree with the graph of $f(x)$. Instead, we can write

$$\lim_{x\to\infty} x^3 - x^2 = \lim_{x\to\infty} x^2(x-1) = \infty$$

as both x^2 and $x - 1$ grow arbitrarily large as $x \to \infty$.

2.6.2 Precise Definitions

Definition. Let f be a function defined on the interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a positive real number M such that

$$\text{if } x > M \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

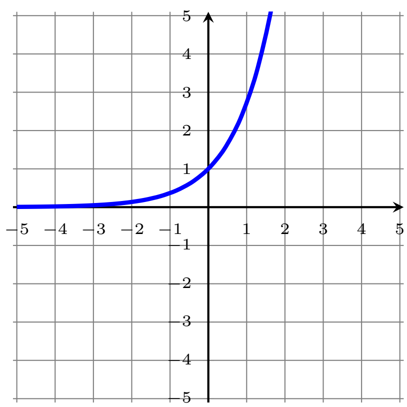
Definition. Let f be a function defined on the interval $(-\infty, a)$. Then

$$\lim_{x \to \infty} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a negative real number N such that

$$\text{if } x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon.$$

Proposition 2.6.11. $\lim_{x \to -\infty} e^x = 0$



This fact is intuitively obvious from the graph or a table of values, but I will include a proof anyway because it's rather simple and I'm bored in my office right now.

Proof. Let $\varepsilon > 0$ and further suppose that $\varepsilon < 1$. We intend to find N such that, $|e^x - 0| < \varepsilon$ whenever $x < N$. We note that the absolute value is unecessary as $0 < e^x$ for every real number x. Choose $N = \ln(\varepsilon)$ and suppose $x < N$. Then

$$|e^x - 0| = e^x < e^N = e^{\ln(\varepsilon)} = \varepsilon$$

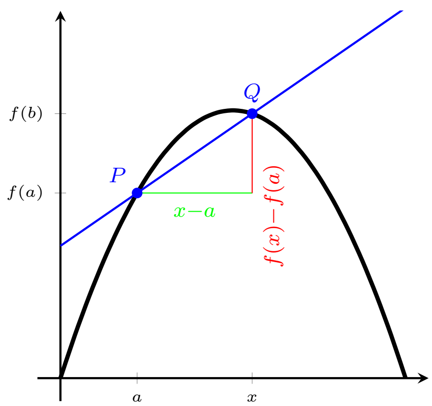
and therefore $\lim_{x \to -\infty} e^x = 0$. $\square$

2.7 Derivatives and Rates of Change

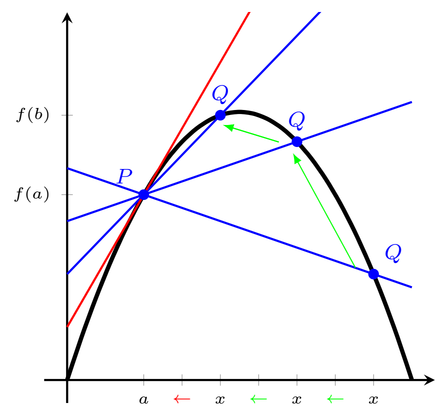
Definition. Given a function f defined on an interval $[a, b]$, the **average rate of change from $x = a$ to $x = b$** is

$$\frac{f(a) - f(b)}{a - b}.$$

Remark. The average rate of change is the slope of the secant line connecting the two points $(a, f(a))$ and $(b, f(b))$.



The secant line.



Secant lines limiting to the tangent line

Definition. Given a function f defined in an interval around a, the **instantaneous rate of change at $x = a$** is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists.

Remark. The instantaneous rate of change tells us the **slope of the tangent line** (to the curve $y = f(x)$) at the point $(a, f(a))$.

Example 2.7.1. Find the equation of the line tangent to the curve $y = -x^2$ at the point $(3, -9)$.

$$\begin{aligned} \lim_{x \to 3} \frac{-x^2 + 3^2}{x - 3} &= \lim_{x \to 3} \frac{-(x-3)(x+3)}{x - 3} \\ &= \lim_{x \to 3} -(x+3) \\ &= -(3+3) = -6. \end{aligned}$$

Using the point-slope form of a line with slope 6 passing through the point $(3, -9)$, we have

$$\begin{aligned} y + 9 &= 6(x - 3) \\ \Rightarrow y &= 6x - 27 \end{aligned}$$

is the equation of the tangent line we wanted.

Definition. The **derivative of a function f at a number a**, denoted $f'(a)$, is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

provided the limit exists.

By making the substitution $h = x - a$, we get the following equivalent definition:

Definition. The **derivative of a function f at a number a**, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

provided the limit exists.

Remark. Here the h represents the distance away from the point a. For reasons that we will see in the next section, this latter definition is the more common of the two definitions of a derivative at a point.

Example 2.7.2. Let $g(x) = \dfrac{2}{x} - 4$. Find $g'(1)$ and $g'(-3)$.

We'll use the first definition to find $g'(1)$ and the second definition to find $g'(-3)$.

$$\begin{aligned} g'(1) &= \lim_{x \to 1} \frac{\left(\frac{2}{x} - 4\right) - \left(\frac{2}{1} - 4\right)}{x - 1} \\ &= \lim_{x \to 1} \frac{\left(\frac{2}{x} - 2\right)}{x - 1} \\ &= \lim_{x \to 1} \frac{\left(\frac{2-2x}{x}\right)}{x - 1} \\ &= \lim_{x \to 1} \frac{-2(x-1)}{x(x-1)} \\ &= \lim_{x \to 1} \frac{-2}{x} \\ &= -2 \end{aligned}$$

and

$$\begin{aligned} g'(-3) &= \lim_{h \to 0} \frac{\left(\frac{2}{-3+h} - 4\right) - \left(\frac{2}{-3} - 4\right)}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{2}{-3+h} + \frac{2}{3}\right)}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{2(3)+2(-3+h)}{3(-3+h)}\right)}{h} \\ &= \lim_{h \to 0} \frac{2h}{h(-9+h)} \\ &= \lim_{h \to 0} \frac{2}{-9+h} \\ &= -\frac{2}{9}. \end{aligned}$$

Example 2.7.3. Find $f'(0)$ for the function $f(x) = \sqrt{1-3x}$.

Using the definition of the derivative at a point a, we have

$$\begin{aligned}
f'(0) &= \lim_{x\to 0} \frac{\sqrt{1-3x}-1}{x} \\
&= \lim_{x\to 0} \frac{\sqrt{1-3x}-1}{x} \cdot \left(\frac{\sqrt{1-3x}+1}{\sqrt{1-3x}+1}\right) \\
&= \lim_{x\to 0} \frac{(1-3x)-1}{x(\sqrt{1-3x}+1)} \\
&= \lim_{x\to 0} \frac{-3x}{x(\sqrt{1-3x}+1)} \\
&= \lim_{x\to 0} \frac{-3}{\sqrt{1-3x}+1} \\
&= \frac{-3}{2}.
\end{aligned}$$

Example 2.7.4. Find $f'(a)$ for the function $f(x) = \sqrt{1-3x}$. [Here we are assuming that $a < \frac{1}{3}$.]

Using the definition of the derivative at a point a, we have

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sqrt{1-3(a+h)}-\sqrt{1-3a}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{1-3(a+h)}-\sqrt{1-3a}}{h} \left(\frac{\sqrt{1-3(a+h)}+\sqrt{1-3a}}{\sqrt{1-3(a+h)}+\sqrt{1-3a}}\right) \\
&= \lim_{h\to 0} \frac{[1-3(a+h)]-[1-3a]}{h\left(\sqrt{1-3(a+h)}+\sqrt{1-3a}\right)} \\
&= \lim_{h\to 0} \frac{-3h}{h\left(\sqrt{1-3(a+h)}+\sqrt{1-3a}\right)} \\
&= \lim_{h\to 0} \frac{-3}{\sqrt{1-3(a+h)}+\sqrt{1-3a}} \\
&= \frac{-3}{2\sqrt{1-3a}}.
\end{aligned}$$

2.8 The Derivative as a Function

Definition. Given a function f, the **derivative of f** is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

The derivative is a function that tells you the slope of the tangent line at every point along the curve. In a physical system, the first derivative of the position function is the *velocity function*.

Remark. There are several equivalent notations for the derivative of $y = f(x)$. They are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x),$$

and we will be using them (notably the first four) interchangeably.

Example 2.8.1. Let $f(x) = \sqrt{x+2}$. Find the derivative $\dfrac{df}{dx}$.

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h \to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h} \left(\frac{\sqrt{(x+h)+2} + \sqrt{x+2}}{\sqrt{(x+h)+2} + \sqrt{x+2}} \right) \\
&= \lim_{h \to 0} \frac{(x+h+2) - (x+2)}{h(\sqrt{(x+h)+2} + \sqrt{x+2})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{(x+h)+2} + \sqrt{x+2})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{(x+h)+2} + \sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}.
\end{aligned}$$

Definition. A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) if it is differentiable at every number in the interval.

Example 2.8.2. Where is $f(x) = |x|$ differentiable? *[Hint: consider separately the cases when $x < 0$, $x = 0$, $x > 0$, and also the limits as $h \to 0^+$ and $h \to 0^-$.]*

It's up to the reader to prove that $f'(x)$ exists when $x \neq 0$. The interesting case happens when $x = 0$. If $f'(0)$ exists, the limit (in the definition of the derivative) must exist. Examining the left and right limits, we see that

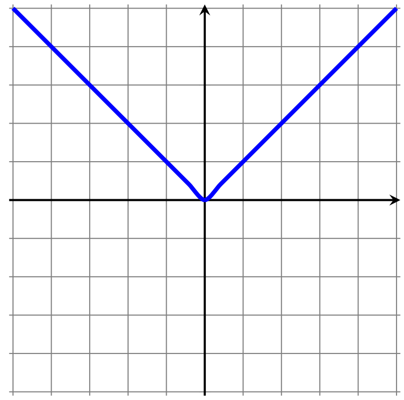
$$\lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

and

$$\lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1,$$

so since the limits do not agree, $f'(0)$ does not exist and thus f is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Below is a graph of $f(x) = |x|$. Notice what the graph of the function looks like at the single point of non-differentiability.



This previous example tells us that functions with *cusps* are not differentiable at these cusps. The following result tells us another way in which a function can fail to be differentiable at a point.

Theorem 2.8.3. *If f is differentiable at a, then f is continuous at a.*

Proof. The proof of this result is an ε-δ argument that we won't go into. Heuristically, it comes down to the fact that the numerator (in the definition of a derivative at a point) implies that $\lim_{x\to a} f(x) = f(a)$; without this, the limit (in the definition of a derivative at a point) simply would not exist. □

Corollary 2.8.4. *If f is discontinuous at a point, f is not differentiable at that point.*

Example 2.8.5. Where is $f(x) = \begin{cases} 2x+7 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$ differentiable?

As we saw in class, if we naïvely took the left and right limits, we might be inclined to say that f is differentiable at 1 as

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = 2.$$

However, $f(x)$ is clearly discontinuous at $x = 1$, Corollary 2.8.4 tells us that f is not differentiable at $x = 1$. For the case when $x < 1$, we have

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[2(x+h)+7] - [2x+7]}{h} = 2,$$

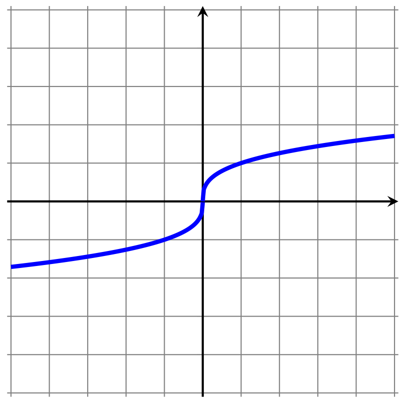
so f is differentiable on $(-\infty, 1)$. Also, when $x > 1$, we have

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = 2x,$$

so f is differentiable on $(1, \infty)$ as well. Therefore f is differentiable on $(-\infty, 1) \cup (1, \infty)$.

There is another way that we can tell graphically if a function is differentiable at a point. If the tangent line is vertical, this corresponds to a derivative that would be ∞ or $-\infty$ (which means the derivative does not exist).

Example 2.8.6. Graph the function $f(x) = \sqrt[3]{x}$. Where, if anywhere, does f fail to be differentiable? Using the definition of the derivative, check your answer.



It looks like there might be a problem at $x = 0$. Indeed

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} \\ &= \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} \\ &= \lim_{x \to 0} \frac{1}{x^{2/3}} = \text{Does Not Exist.} \end{aligned}$$

So $f(x) = \sqrt[3]{x}$ is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Definition. The **second derivative of** f is the function $f'' = (f')'$. It is the derivative of the derivative f'. In Leibniz notation, we write $\dfrac{d^2 f}{dx^2}$ or $\dfrac{d^2 y}{dx^2}$.

The **third derivative of** f is the function $f''' = (f'')'$. It is the derivative of the second derivative f''. In Leibniz notation, we write $\dfrac{d^3 f}{dx^3}$ or $\dfrac{d^3 y}{dx^3}$.

The n^{th} **derivative of** f is the function $f^{(n)} = (f^{(n-1)})'$. It is the derivative of the $(n-1)^{\text{st}}$ derivative $f^{(n-1)}$. In Leibniz notation, we write $\dfrac{d^n f}{dx^n}$ or $\dfrac{d^n y}{dx^n}$.

Remark. In a physical system, the first derivative of the position function is the *velocity* function. The second derivative of the position function is the *acceleration* function. The third derivative is the *jerk* function. The fourth derivative is the *snap* function. The fifth derivative is the *crackle* function. The sixth derivative is the *pop* function.

Example 2.8.7. Find the fourth derivative $f^{(4)}(t)$ of the function $f(t) = t^4$.

First we find $f'(t)$:

$$\begin{aligned} f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} &= \lim_{h\to 0} \frac{(t+h)^4 - t^4}{h} \\ &= \lim_{h\to 0} \frac{4t^3h + 6t^2h^2 + 4th^3 + h^4}{h} \\ &= \lim_{h\to 0} 4t^3 + 6t^2h + 4th^2t + h^3 \\ &= 4t^3. \end{aligned}$$

Now we find $f''(t)$:

$$\begin{aligned} f''(t) = \lim_{h\to 0} \frac{f'(t+h) - f'(t)}{h} &= \lim_{h\to 0} \frac{4(t+h)^3 + 4t^3}{h} \\ &= \lim_{h\to 0} \frac{12t^2h + 12th^2}{h} \\ &= \lim_{h\to 0} 12t^2 + 12th \\ &= 12t^2. \end{aligned}$$

Then we find $f'''(t)$:

$$\begin{aligned} f'''(t) = \lim_{h\to 0} \frac{f''(t+h) - f''(t)}{h} &= \lim_{h\to 0} \frac{12(t+h)^2 - 12t^2}{h} \\ &= \lim_{h\to 0} \frac{12th}{h} \\ &= \lim_{h\to 0} 12t \\ &= 12t. \end{aligned}$$

Finally we find $f^{(4)}(t)$:

$$\begin{aligned} f^{(4)}(t) = \lim_{h\to 0} \frac{f'''(t+h) - f'''(t)}{h} &= \lim_{h\to 0} \frac{12(t+h) - 12t}{h} \\ &= \lim_{h\to 0} \frac{12h}{h} \\ &= \lim_{h\to 0} 12 \\ &= 12. \end{aligned}$$

3 Differentiation Rules

3.1 Derivatives of Polynomials and Exponential Functions

As we saw previously, finding derivatives by taking limits is an absolute nightmare. Thankfully, there are some general patterns that arise that will make finding derivatives much faster for us.

Proposition 3.1.1 (Derivative of a Constant)**.** *Let c be any real number. Then*

$$\frac{d}{dx}[c] = 0.$$

Proof. The result should be intuitive as the slope of the tangent line $y = c$ is 0 at every point. The proof is left as a very simple exercise. Just use the limit definition of a derivative. □

Proposition 3.1.2 (Power Rule for Derivatives)**.** *Let n be any real number. Then*

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

We can see from Example 2.8.7 that this seems to be true (and indeed it is, I promise). When n is a nonnegative integer, the proof is again straightforward (although clunky given that you have to expand the binomial $(x+h)^n$).

Proposition 3.1.3 (Constant Multiple Rule for Derivatives)**.** *Let c be any real number and $f(x)$ any function. Then*

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}[f(x)].$$

Proof. The proof of this result follows from the constant multiple rule of limits. □

Proposition 3.1.4 (Sum/Difference Rule for Derivatives)**.** *If $f(x)$ and $g(x)$ are both differentiable, then*

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)].$$

Proof. The proof of this result follows from the sum/difference rules for limits. □

Remark. Note that the product and quotient rules for derivatives do not behave as nicely as you might expect. We'll address these in a future lecture.

With these new rules, it now becomes very quick to find derivatives of things like polynomials and rational functions.

Example 3.1.5. Find $\dfrac{df}{dx}$ where $f(x) = x^{31} - 27x^2 + 18x + 1$.

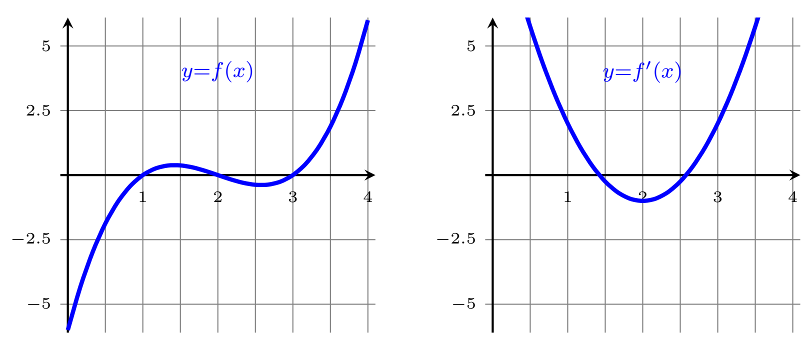
$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}\left[x^{31} - 27x^2 + 18x + 1\right] \\
&= \frac{d}{dx}\left[x^{31}\right] - \frac{d}{dx}\left[27x^2\right] + \frac{d}{dx}\left[18x\right] + \frac{d}{dx}[1] && \text{(sum/difference rule)} \\
&= \frac{d}{dx}\left[x^{31}\right] - 27\frac{d}{dx}\left[x^2\right] + 18\frac{d}{dx}\left[x\right] + \frac{d}{dx}[1] && \text{(constant multiple rule)} \\
&= 31x^{30} - 54x + 18 && \text{(power rule)}
\end{aligned}$$

Example 3.1.6. Find $\dfrac{dg}{dt}$ where $g(t) = \dfrac{t^{14} + 2t^7 + 13t^3 + 1}{t^5}$.

$$\begin{aligned}
\frac{dg}{dt} &= \frac{d}{dt}\left[\frac{t^{14} + 2t^7 + 13t^3 + 1}{t^5}\right] \\
&= \frac{d}{dt}\left[t^9 + 2t^2 + 13t^{-2} + t^{-5}\right] \\
&= \frac{d}{dt}\left[t^9\right] + \frac{d}{dt}\left[2t^2\right] + \frac{d}{dt}\left[13t^{-2}\right] + \frac{d}{dt}\left[t^{-5}\right] \\
&= 9t^8 + 4t - 26t^{-3} - 5t^{-6}
\end{aligned}$$

Example 3.1.7. Given $f(x) = x^3 - 6x^2 + 11x - 6$, sketch a graph f and f'.

First we find f'. By the power rule and sum/difference rules, we have that $f'(x) = 3x^2 - 12x + 11$. Notice the relationship between points with horizontal tangents in $f(x)$ correspond to x-intercepts of $f'(x)$. We also have a correspondence between positive (*resp.* negative) slopes of tangent lines of $f(x)$ with positive (*resp.* negative) y-values of $f'(x)$. This means that, given a graph of a function and its derivative, we should be able to determine which is which.



3.1.1 Exponential Functions

Recall that the **exponential function f with base b** is

$$f(x) = b^x,$$

where $b > 0$ and $b \neq 1$. Exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, and they are continuous for all positive real numbers.

Remark. We could allow $b = 1$ in the definition above, but then $f(x) = 1^x = 1$, which is a constant function, so its behavior is both uninteresting and fundamentally different from all other base values, so we usually ignore it.

Proposition 3.1.8 (Derivative of an Exponential)**.** *Let b be any positive number except 1. Then*

$$\frac{d}{dx}[b^x] = b^x \ln(b).$$

Partial proof. Let $f(x) = b^x$. Then

$$f'(x) = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} = b^x \cdot \left[\lim_{h\to 0} \frac{b^h - 1}{h}\right]$$

and this last limit is exactly $f'(0)$, hence

$$f'(x) = b^x \cdot f'(0).$$

One can numerically approximate $f'(0)$ for any value b and see that $f'(0) = \ln(b)$. This isn't a coincidence, but we'll wait until a later section to see this more explicitly. □

For computational purposes, it would be convenient to use the base for which $f'(0) = 1$. Indeed one sees that this is the number e, hence

Proposition 3.1.9 (Derivative of the Natural Exponential)**.**

$$\frac{d}{dx}[e^x] = e^x.$$

3.2 The Product and Quotient Rules

From last time we had some nice properties of derivatives - we could differentiate across scalar multiplication and addition (for this reason, we say that $\frac{d}{dx}$ is a "linear operator"). However, as we will see, derivatives of products and quotients do not behave quite as obviously as we might hope.

Theorem 3.2.1 (Product Rule)**.** *Let f and g be differentiable functions. Then*

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Proof. Using the limit definition of the derivative, we have

$$\begin{aligned}
&\frac{d}{dx}[f(x)g(x)] \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h) - g(x)]}{h} \\
&= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] \cdot \left[\lim_{h\to 0} g(x+h)\right] + \left[\lim_{h\to 0} f(x)\right] \cdot \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

□

Example 3.2.2. Let $h(x) = (2x^3 + 7)(x - 5\sqrt{x})$. Find $h'(x)$ using the product rule. Check your answer by first expanding out the function (FOIL) and then taking the derivative.

We first identify two functions $f(x) = 2x^3 + 7$ and $g(x) = x - 5\sqrt{x} = x - 5x^{1/2}$ so that $h(x) = f(x)g(x)$. Now,

$$f'(x) = \frac{d}{dx}[2x^3 + 7] = 2\frac{d}{dx}[x^3] + \frac{d}{dx}[7] = 2[3x^2] + 0 = 6x^2,$$

and

$$g'(x) = \frac{d}{dx}[x - 5x^{1/2}] = \frac{d}{dx}[x] - 5\frac{d}{dx}[x^{1/2}] = 1 - 5\left[\frac{1}{2}x^{-1/2}\right] = 1 - \frac{5}{2}x^{-1/2}.$$

Thus, by the product rule,

$$h'(x) = 6x^2\left(x - 5x^{1/2}\right) + \left(2x^3 + 7\right)\left(1 - \frac{5}{2}x^{-1/2}\right).$$

Unsurprisingly, the quotients of differentiable functions do not behave as obviously as we might like them to either.

Theorem 3.2.3 (Quotient Rule)**.** *Let f and g be differentiable functions. Then for any x where $g(x) \neq 0$,*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof. The proof of this theorem uses a similar trick as in the proof of the product rule: add $f(x)g(x) - f(x)g(x)$ in the numerator, then split up the limits. It is left as an exercise to the reader. □

Example 3.2.4. Let $s(t) = \dfrac{5}{t^3 - 9}$. Find $s'(2)$.

Let $f(t) = 5$ and $g(t) = t^3 - 9$ so that $s(t) = \dfrac{f(t)}{g(t)}$. Then

$$f'(t) = 0 \quad \text{and} \quad g'(t) = 3t^2.$$

So by the quotient rule,

$$s'(t) = \frac{0(t^3 - 9) - 5(3t^2)}{(t^3 - 9)^2} = \frac{-15t^2}{(t^3 - 9)^2}$$

and thus

$$s'(2) = -\frac{15(2)^2}{[(2)^3 - 9]^2} = -60.$$

Example 3.2.5. Let f and g be differentiable functions, and let $F(x) = f(x)g(x)$ and $G(x) = \dfrac{f(x)}{g(x)}$. Suppose $f(-1) = 5$, $f'(-1) = 12$, $g(-1) = -3$, and $g'(-1) = 8$. Find $F'(-1)$ and $G'(-1)$.

The product rule tells us that

$$F'(-1) = f'(-1)g(-1) + f(-1)g'(-1) = (12)(-3) + (5)(8) = -36 + 40 = 4.$$

The quotient rule tells us that

$$G'(-1) = \frac{f'(-1)g(-1) - f(-1)g'(-1)}{[g(-1)]^2} = \frac{(12)(-3) - (5)(8)}{[-3]^2} = -\frac{76}{9}.$$

Example 3.2.6. Let $f(x) = xe^x$. Compute the first three derivatives of f and use this to find the n^{th} derivative $f^(n)$.

By the product rule

$$\begin{aligned}
f'(x) &= e^x + xe^x \\
f''(x) &= e^x + e^x + xe^x = 2e^x + xe^x \\
f'''(x) &= e^x + e^x + e^x + xe^x = 3e^x + xe^x \\
&\vdots \\
f^{(n)}(x) &= ne^x + xe^x.
\end{aligned}$$

3.3 Derivatives of Trigonometric Functions

Lemma 3.3.1.

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$$

Proof. Recall from Section 2.3 that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Then

$$\begin{aligned}
\lim_{x \to 0} \frac{\cos(x) - 1}{x} = \lim_{x \to 0} \left(\frac{\cos(x) - 1}{x} \right) \left(\frac{\cos(x) + 1}{\cos(x) + 1} \right) &= \lim_{x \to 0} \frac{\cos^2(x) - 1}{x(\cos(x) + 1)} \\
&= \lim_{x \to 0} \frac{-\sin^2(x)}{x(\cos(x) + 1)} \\
&= \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{-\sin(x)}{\cos(x) + 1} \\
&= \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \cdot \left(\lim_{x \to 0} \frac{-\sin(x)}{\cos(x) + 1} \right) \\
&= 1 \cdot 0 = 0.
\end{aligned}$$

□

With these two lemmas, we can prove the following result.

Proposition 3.3.2 (Derivative of Sine/Cosine)**.**

$$\frac{d}{dx}[\sin(x)] = \cos(x) \quad \textit{and} \quad \frac{d}{dx}[\cos(x)] = -\sin(x).$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \frac{[\sin(x)\cos(h) + \sin(h)\cos(x)] - \sin(x)}{h} && \text{(angle sum/difference identity)} \\
&= \lim_{h \to 0} \frac{\sin(x)[\cos(h) - 1] + \sin(h)\cos(x)}{h} \\
&= \lim_{h \to 0} \sin(x) \frac{\cos(h) - 1}{h} + \lim_{h \to 0} \frac{\sin(h)}{h} \cos(x) \\
&= \sin(x)(0) + (1)\cos(x) && \text{(applying Lemma ??)} \\
&= \cos(x).
\end{aligned}$$

And similarly,

$$\begin{aligned}
\frac{d}{dx}[\cos(x)] &= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{[\cos(x)\cos(h)-\sin(x)\sin(h)]-\cos(x)}{h} && \text{(angle sum/difference identity)} \\
&= \lim_{h\to 0} \frac{\cos(x)[\cos(h)-1]-\sin(h)\sin(x)}{h} \\
&= \lim_{h\to 0} \cos(x)\frac{\cos(h)-1}{h} - \lim_{h\to 0} \frac{\sin(h)}{h}\sin(x) \\
&= \cos(x)(0) - (1)\sin(x) \\
&= -\sin(x).
\end{aligned}$$

□

Since $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, $\csc(x) = \dfrac{1}{\sin(x)}$, $\sec(x) = \dfrac{1}{\cos(x)}$, and $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, we can now use the quotient rule to complete our list of derivatives of trigonometric functions.

Proposition 3.3.3 (Derivatives of Trigonometric Functions)**.**

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \cos(x) & \qquad \frac{d}{dx}[\cos(x)] &= -\sin(x) \\
\frac{d}{dx}[\sec(x)] &= \sec(x)\tan(x) & \qquad \frac{d}{dx}[\csc(x)] &= -\csc(x)\cot(x) \\
\frac{d}{dx}[\tan(x)] &= \sec^2(x) & \qquad \frac{d}{dx}[\cot(x)] &= -\csc^2(x)
\end{aligned}$$

Proof. We'll obtain the derivative of $\tan(x)$, and leave the remaining derivatives as an exercise for the reader.

Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$ so that $\tan(x) = \dfrac{\sin(x)}{\cos(x)} = \dfrac{f(x)}{g(x)}$. Then

$$f'(x) = \cos(x) \quad \text{and} \quad g'(x) = -\sin(x).$$

Thus, by the quotient rule

$$\begin{aligned}
\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] &= \frac{\cos(x)\cos(x)-\sin(x)[-\sin(x)]}{\cos^2(x)} \\
&= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = \sec^2(x). && \text{(since } \cos^2\theta+\sin^2\theta = 1\text{)}
\end{aligned}$$

□

Example 3.3.4. Find $\frac{dh}{dx}$ where $h(x) = 7x^2 \left[2\tan(x) + 3\sec(x)\right]$.

Again, let $f(x) = 7x^2$ and $g(x) = 2\tan(x) + 3\sec(x)$. Then

$$f'(x) = 14x \quad \text{and} \quad g'(x) = 2\sec^2(x) + 3\sec(x)\tan(x).$$

So, by the product rule, we have that

$$h'(x) = f'(x)g(x) + f(x)g'(x) = 14x[2\tan(x) + 3\sec(x)] + 7x^2\left[2\sec^2(x) + 3\sec(x)\tan(x)\right].$$

Example 3.3.5. Find $f'(x)$ for $f(x) = \dfrac{x^3\sin(x)}{x+1}$.

Notice that our numerator is a product of functions, so we're going to have to apply a product rule within the quotient rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{x^3\sin(x)}{x+1}\right] \\ &= \frac{\frac{d}{dx}[x^3\sin(x)](x+1) - x^3\sin(x)\frac{d}{dx}[x+1]}{(x+1)^2} && \text{(quotient rule)} \\ &= \frac{\left(\frac{d}{dx}[x^3]\sin(x) + x^3\frac{d}{dx}[\sin(x)]\right)(x+1) - x^3\sin(x)\frac{d}{dx}[x+1]}{(x+1)^2} && \text{(product rule)} \\ &= \frac{3x^2\sin(x)(x+1) + x^3\cos(x)(x+1) - x^3\sin(x)}{(x+1)^2}. \end{aligned}$$

Example 3.3.6. Find $f'(\theta)$ for $f(\theta) = \sin^2\theta$.

$$\begin{aligned} f'(\theta) = \frac{d}{d\theta}[\sin\theta\sin\theta] &= \frac{d}{d\theta}[\sin\theta]\sin\theta + \sin\theta\frac{d}{d\theta}[\sin\theta] && \text{(product rule)} \\ &= \cos\theta\sin\theta + \sin\theta\cos\theta = 2\sin\theta\cos\theta. \end{aligned}$$

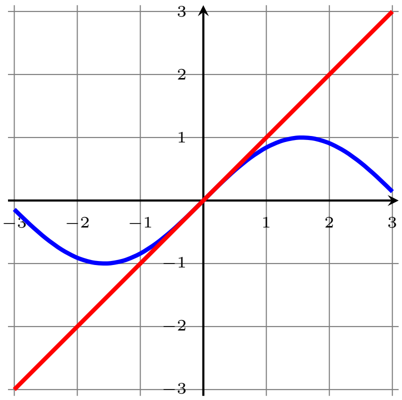
Example 3.3.7. Find $f'(\theta)$ for $f(\theta) = \sin^3\theta$.

$$\begin{aligned} f'(\theta) = \frac{d}{d\theta}\left[\sin^3\theta\right] &= \frac{d}{d\theta}\left[\sin\theta\sin^2\theta\right] \\ &= \frac{d}{d\theta}[\sin\theta]\sin^2\theta + \sin\theta\frac{d}{d\theta}\left[\sin^2\theta\right] \\ &= \frac{d}{d\theta}[\sin\theta]\sin^2\theta + \sin\theta\frac{d}{d\theta}[\sin\theta\sin\theta] \\ &= \frac{d}{d\theta}[\sin\theta]\sin^2\theta + \sin\theta\left(\frac{d}{d\theta}[\sin\theta]\sin\theta + \sin\theta\frac{d}{d\theta}[\sin\theta]\right) \\ &= \cos\theta\sin^2\theta + \sin\theta\left(\cos\theta\sin\theta + \sin\theta\cos\theta\right) \\ &= 3\sin^2\theta\cos\theta. \end{aligned}$$

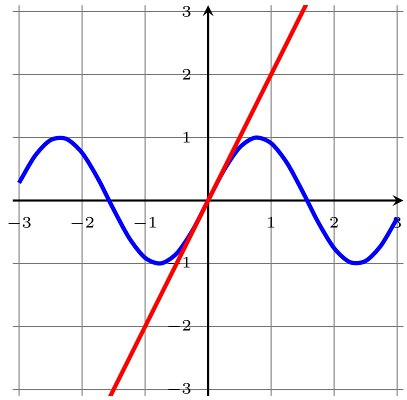
There appears to be a pattern forming here. We conjecture that $\dfrac{d}{d\theta}[\sin^n\theta] = n\sin^{n-1}\theta\cos\theta$. Indeed, if we think about $\sin^n(x) = f(g(x))$ where $f(x) = x^n$ and $g(x) = \sin(x)$, we see that somehow it's like there's a combination of the derivatives $f'(x) = nx^{n-1}$ and $g'(x) = \cos(x)$ involved in the derivative of $f(g(x))$. We'll make this formal in the next section.

3.4 The Chain Rule

If one looks at the tangent lines of $\sin(x)$ and $\sin(2x)$ at $x = 0$, then it is clear that the tangent line for $\sin(x)$ has slope 1, and the tangent line for $\sin(2x)$ has slope 2.



Graph of $y = \sin(x)$



Graph of $y = \sin(2x)$

Notice also that 1 and 2 are the derivatives of x and $2x$, respectively. It seems then that the derivative of a function might also depend on the derivative of the argument of that function.

What if you were asked to find the derivative of a generic composite function $f(g(x))$? Certainly you could approach with limits, but limits are extremely messy and it'd be much nicer if we had a rule that gave us an all-inclusive approach to composite functions. Indeed, there is such a rule:

Theorem 3.4.1 (Chain Rule)**.** *Suppose f and g are both differentiable functions. Then the composite $(f \circ g)(x) = f(g(x))$ is differentiable and the derivative is given by*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, letting $y = f(u)$ and $u = g(x)$, we have that $y = f(g(x))$ and so we would write

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

We'll prove the above theorem in the case of functions that are relatively well-behaved; the result is true in general, but requires substantially more effort to prove.

Proof. We aim to show that $f \circ g$ is differentiable at an arbitrary point a. Suppose we can find a small interval around a so that $g(x) \neq g(a)$ for all x in this interval (for sufficiently well-behaved functions, this is completely reasonable). Then

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&= \lim_{x \to a} \left(\frac{f(g(x)) - f(g(a))}{x - a}\right) \cdot \left(\frac{g(x) - g(a)}{g(x) - g(a)}\right) \\
&= \lim_{x \to a} \left(\frac{f(g(x)) - f(g(a))}{g(x) - g(a)}\right) \cdot \left(\frac{g(x) - g(a)}{x - a}\right) \\
&= \left[\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}\right] \cdot \left[\lim_{x \to a} \frac{g(x) - g(a)}{x - a}\right].
\end{aligned}$$

Since g is continuous at a, then $g(x) \to g(a)$ as $x \to a$, so we can rewrite the first limit in the product above

$$\begin{aligned}(f \circ g)'(a) &= \left[\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}\right] \cdot \left[\lim_{x\to a} \frac{g(x) - g(a)}{x - a}\right] \\ &= \left[\lim_{g(x)\to g(a)} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}\right] \cdot \left[\lim_{x\to a} \frac{g(x) - g(a)}{x - a}\right] \\ &= f'(g(a)) \cdot g'(a).\end{aligned}$$

□

Remark. When it comes to the chain rule, often times the most difficult part is determining what the two functions that form your composite function are.

Example 3.4.2. Find the derivative of $h(x) = \tan(3x)$

Let $f(x) = \tan(x)$ and $g(x) = 3x$ so that $h(x) = f(g(x))$. Then

$$f'(x) = \sec^2(x)$$

and

$$g'(x) = 3,$$

so

$$h'(x) = f'(g(x)) \cdot g'(x) = \sec^2(g(x)) \cdot 3 = 3\sec^2(3x).$$

Example 3.4.3. Find $\dfrac{dh}{dt}$ where $h(t) = (t^2 - 7)^{861}$.

Let $f(t) = t^{861}$ and $g(t) = (t^2 - 7)$ so that $h(t) = f(g(t))$. Then

$$f'(t) = 861t^{860}$$

and

$$g'(t) = 2t,$$

so

$$h'(t) = f'(g(t)) \cdot g'(t) = 861(t^2 - 7)^{860} \cdot 2t.$$

Sometimes you may need to use the chain rule in conjunction with the product or quotient rules.

Example 3.4.4. Find $r'(\theta)$ where $r(\theta) = \sqrt{(\theta+1)\sin\theta}$.

Let $f(\theta) = \sqrt{\theta}$, and $g(\theta) = (\theta+1)\sin\theta$. Then

$$f'(\theta) = \frac{1}{2}\theta^{-1/2}$$

and, using the product rule, we have

$$g'(\theta) = \sin\theta + (\theta+1)\cos\theta,$$

so using the chain rule

$$r'(\theta) = f'(g(\theta)) \cdot g'(\theta) = \frac{1}{2}\left[(\theta+1)\sin\theta\right]^{-1/2}\left[\sin\theta + (\theta+1)\cos\theta\right].$$

Example 3.4.5. Find the derivative of $F(x) = \dfrac{\cos(\sqrt{x})}{\csc(x)}$.

Let $f(x) = \cos(x)$, $g(x) = \sqrt{x}$ and $h(x) = \csc(x)$. Then the numerator is $f(g(x))$ and the denominator is $h(x)$. We have that

$$f'(x) = -\sin(x)$$

and

$$g'(x) = \frac{1}{2}x^{-1/2},$$

so

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = -\frac{1}{2}\cos(\sqrt{x})x^{-1/2}.$$

We also have that

$$h'(x) = -\csc(x)\cot(x),$$

so using the quotient rule,

$$\begin{aligned} F'(x) &= \frac{(f \circ g)'(x)h(x) - (f \circ g)(x)h'(x)}{[h(x)]^2} \\ &= \frac{-\frac{1}{2}\cos(\sqrt{x})x^{-1/2}\csc(x) + \cos(\sqrt{x})\csc(x)\cot(x)}{\csc^2(x)}. \end{aligned}$$

Sometimes, we may even have to use an embedded chain rule *(chain rule-ception)*.

Example 3.4.6. Find the derivative $\dfrac{dT}{d\varphi}$ of $T(\varphi) = \sin(\tan(\csc\varphi))$. Let $f(\varphi) = \sin\varphi$, $g(\varphi) = \tan\varphi$, and $h(\varphi) = \csc\varphi$, so then $T(\varphi) = f(g(h(\varphi))$. We have that

$$f'(\varphi) = \cos\varphi, \qquad g'(\varphi) = \sec^2\varphi, \qquad \text{and } h'(\varphi) = -\csc\varphi\cot\varphi.$$

So then

$$\begin{aligned}
T'(\varphi) &= f'(g(h(\varphi))) \cdot (g \circ h)'(\varphi) \\
&= f'(g(h(\varphi)) \cdot g'(h(\varphi)) \cdot h'(\varphi) \\
&= \cos(\tan(\csc\varphi)) \cdot \sec^2(\csc\varphi) \cdot [-\csc\varphi\cot\varphi] \\
&= -\cos(\tan(\csc\varphi))\sec^2(\csc\varphi)\csc\varphi\cot\varphi
\end{aligned}$$

Example 3.4.7. Find all points on the graph of $y = 1+\sqrt{8x^2 - x^4}$ where the tangent line is horizontal. Confirm your results by sketching a graph.

The slope of a horizontal tangent line is 0, so we're looking for places where $y' = 0$. Let $f(x) = 1 + \sqrt{x}$ and $g(x) = 8x^2 - x^4$ so that $y = f(g(x))$. Then

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

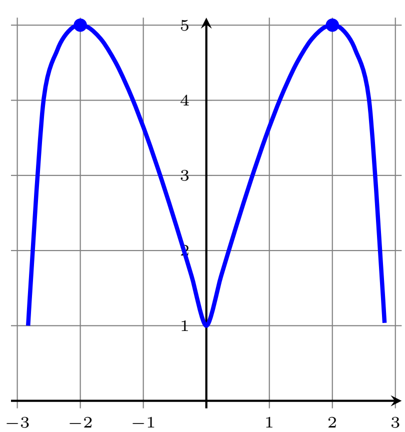
and

$$g'(x) = 16x - 4x^3,$$

so

$$y' = f'(g(x)) \cdot g'(x) = \frac{16x - 4x^3}{2\sqrt{8x^2 - x^4}} = \frac{4x(4 - x^2)}{2\sqrt{4x^2 - x^4}}.$$

We see that $y' = 0$ when $x = 0, \pm 2$. However, 0 is not in the domain of y', so in fact the only horizontal tangent lines occur when $x = \pm 2$.



3.5 Implicit Differentiation

Every function we've encountered up to this point can be described as one variable *explicilty* in terms of another variable, for example

$$y = \sqrt{x-1}$$
$$y = 47x^2\sin(x)$$

However, there are some functions that may be defined *implicitly* by a relation between x and y, for example

$$x^2 + y^2 = 169 \qquad \text{(circle of radius 13)}$$
$$x = \frac{2}{3}y^2 \qquad \text{(parabola opening to the right)}$$
$$4(x^2+y^2) = (x^2+y^2-2xy)^2 \qquad \text{(cardioid)}$$

In some of these cases, it's easy to represent the implicit function as at least one explicit function, but in general that need not happen (as is the case with the cardioid, which requires a minimum of four explicit functions). In these cases, we would still like to be able to find the derivative $\frac{dy}{dx}$, say to find the equation of the tangent line. The key is to use the chain rule and treat $y = y(x)$ as a function of x.

Example 3.5.1. If $x^2 + y^2 = 400$, find $\dfrac{dy}{dx}$.

Since we're treating y as a function of x, we actually have that $y^2 = [y(x)]^2 = g(y(x))$, where $g = x^2$. Since this is a composite function, we'll need to use the chain rule. Indeed, we have that

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[g(y(x))] = g'(y(x))\cdot y'(x) = g'(y(x))\frac{dy}{dx} = 2[y(x)]\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Since equal functions have equal derivatives everywhere, we can take a derivative of both sides of our given relation

$$x^2 + y^2 = 400$$
$$\frac{d}{dx}\left[x^2+y^2\right] = \frac{d}{dx}[400]$$
$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = 0$$
$$2x + 2y\frac{dy}{dx} = 0$$

and now we can solve for $\frac{dy}{dx}$ as we would for any other variable

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Example 3.5.2. Find y' implicitly for the equation $y = \sin(xy)$.

Taking a derivative of both sides of the given equation (with respect to x), we get

$$\begin{aligned}
\frac{d}{dx}[y] &= \frac{d}{dx}[\sin(xy)] \\
y' &= \cos(xy)\frac{d}{dx}[xy] && \text{(chain rule)} \\
y' &= \cos(xy)\,(y + xy') && \text{(product rule)} \\
y' &= y\cos(xy) + x\cos(xy)y' \\
-y\cos(xy) &= x\cos(xy)y' - y' \\
-y\cos(xy) &= (x\cos(xy) - 1)y' \\
\Rightarrow \quad y' &= \frac{-y\cos(xy)}{x\cos(xy) - 1}.
\end{aligned}$$

Example 3.5.3. Given the relation $x^2 - y^2 = 81$, find the second derivative $\frac{d^2y}{dx^2}$.

We begin by finding the first derivative

$$\begin{aligned}
\frac{d}{dx}\left[x^2 - y^2\right] &= \frac{d}{dx}[81] \\
2x - 2y\frac{dy}{dx} &= 0 \\
\Rightarrow \quad \frac{dy}{dx} &= \frac{x}{y}. && (3.5.1)
\end{aligned}$$

Now we take the derivative of each side of this new equation (with respect to x)

$$\begin{aligned}
\frac{d}{dx}\left[\frac{dy}{dx}\right] &= \frac{d}{dx}\left[\frac{x}{y}\right] \\
\frac{d^2y}{dx^2} &= \frac{y - x\frac{dy}{dx}}{y^2}. && (3.5.2)
\end{aligned}$$

Now, we're not quite done yet as we want to represent the second derivative entirely in terms of x and y. So, we substitute Equation 3.5.1 into Equation 3.5.2 and get

$$\frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y^2 - x}{y^3}.$$

Example 3.5.4. Find the equation of the tangent lines to the cardioid given by $4(x^2+y^2) = (x^2+y^2-2x)^2$ at the point $(0,-2)$.

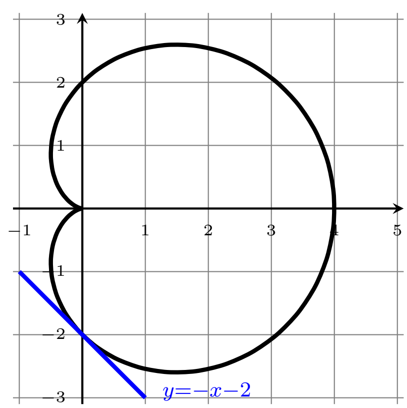
To find the horizontal tangent lines, we use implicit differentiation to find $\frac{dy}{dx}$ and then plug in $(0,-2)$ to find the slope of the tangent line. Taking a derivative of both sides the given equation (with respect to x), we get

$$\begin{aligned}\frac{d}{dx}\left[4(x^2+y^2)\right] &= \frac{d}{dx}\left[\left(x^2+y^2-2x\right)^2\right]\\ 4\left(2x+2y\frac{dy}{dx}\right) &= 2\left(x^2+y^2-2x\right)\frac{d}{dx}\left[x^2+y^2-2x\right]\\ 4\left(2x+2y\frac{dy}{dx}\right) &= 2\left(x^2+y^2-2x\right)\left(2x+2y\frac{dy}{dx}-2\right)\end{aligned}$$

Rather than solve for $\frac{dy}{dx}$ in general, it may be a little we can substitute $(x,y)=(0,-2)$ and then solve for $\frac{dy}{dx}$ (and this should be a little less clunky).

$$\begin{aligned}4\left(2(0)+2(-2)\frac{dy}{dx}\right) &= 2\left((0)^2+(-2)^2-2(0)\right)\left(2(0)+2(-2)\frac{dy}{dx}-2\right)\\ -16\frac{dy}{dx} &= 8\left(-4\frac{dy}{dx}-2\right)\\ \frac{dy}{dx} &= -1\end{aligned}$$

and thus the equation of our tangent line through $(0,-2)$ is $y=-x-2$.



3.5.1 Derivatives of Inverse Trigonometric Functions

Theorem 3.5.5. *All six inverse trigonometric functions are differentiable and their derivatives are*

$$\frac{d}{dx}\left[\sin^{-1}x\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\csc^{-1}x\right] = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\cos^{-1}x\right] = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\sec^{-1}x\right] = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\tan^{-1}x\right] = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\cot^{-1}x\right] = -\frac{1}{1+x^2}$$

Proof. Certainly these functions are differentiable since their inverses are differentiable, so we will just find the derivative of $y = \cos^{-1} x$ and leave the remaining five derivatives as an exercise. Since $y = \cos^{-1} x$, we have that $\cos y = x$. Implicitly differentiating this equation yields

$$\begin{aligned}\frac{d}{dx}[\cos y] &= \frac{d}{dx}[x] \\ -\sin y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin y}.\end{aligned}$$

Now, by the pythagorean identity, $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$, so

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

□

Remark. The derivatives for $\sec^{-1}$ and $\csc^{-1}$ may differ slightly depending on the domain chosen for these functions. We'll assume the standard domains on which each of these functions are continuous.

Example 3.5.6. Let $y = \arcsin(2x)$. Find $\frac{dy}{dx}$. From Theorem 3.5.5 and the chain rule, it follows very quickly that we have

$$y' = \frac{2}{\sqrt{1 - (2x)^2}} = \frac{2}{\sqrt{1 - 4x^2}}.$$

Example 3.5.7. Find $f'\left(\frac{1}{2}\right)$ where $f(x) = 3\arccos(x^2)$. Once again, from Theorem 3.5.5 and the chain rule, we have

$$f'(x) = -\frac{6x}{\sqrt{1 - (x^2)}} = -\frac{6x}{\sqrt{1 - x^4}}.$$

So, when $x = \frac{1}{2}$, we get

$$f'\left(\frac{1}{2}\right) = -\frac{3}{\sqrt{1 - 1/16}} = -4\sqrt{\frac{3}{5}} \approx -3.09839.$$

Example 3.5.8. Find y'' where $y = \arctan(x)$. From Theorem 3.5.5, we have

$$y' = \frac{1}{1 + x^2}.$$

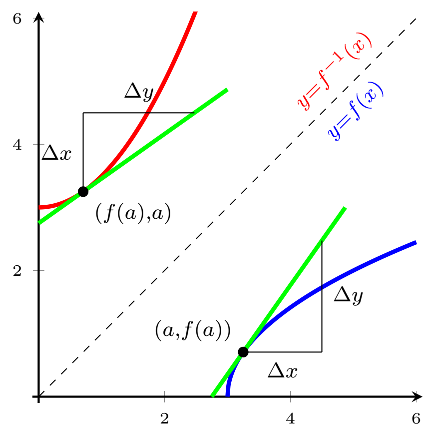
Differentiating with respect to x a second time and applying the chain rule yields

$$y'' = -\frac{2x}{(1 + x^2)^2}.$$

In this author's own experience, the $\arcsin x$ and $\arctan x$ functions tend to appear most often in the real world because $\sin x$ and $\tan x$ are incredibly common.

3.5.2 Calculus of Inverse Functions

From what we know about continuous functions, we have that, if f is continuous and one-to-one, then its inverse should be as well (and indeed it is). What's more, if f is differentiable at a point c and $f'(a) \neq 0$, then we expect that is f^{-1} is also differentiable at $f(a)$.



The picture above even seems to suggest that there is a correspondence between the slope of the tangent line of f at a with the slope of the tangent line of f^{-1} at $f(a)$. The slope of the tangent line of f at a is $m = \frac{\Delta y}{\Delta x}$ and the slope of the tangent line of f^{-1} at $f(a)$ is $\frac{\Delta x}{\Delta y} = \frac{1}{m}$. Indeed, this is always true, and it is shown in the following theorem.

Theorem 3.5.9 (Inverse Function Theorem)**.** *Suppose f is a one-to-one differentiable function with inverse f^{-1} and suppose that $f(a) = b$. If $f'(a) \neq 0$, then*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Proof. Let $y = f(x)$ and $b = f(a)$, so that $f^{-1}(y) = x$ and $f^{-1}(b) = a$. By continuity, as $y \to b$, then $f^{-1}(y) = x \to f^{-1}(b) = a$.

$$\begin{aligned}
\left(f^{-1}\right)'(b) &= \lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} \\
&= \lim_{x \to a} \frac{x - a}{f(x) - f(a)} \\
&= \lim_{x \to a} \frac{1}{\frac{f(x)-f(a)}{x-a}} \\
&= \frac{1}{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}} \\
&= \frac{1}{f'(a)} \\
&= \frac{1}{f'(f^{-1}(b))}.
\end{aligned}$$

□

Why is this useful? Well, it allows us to find the derivative of the inverse at a point *without having to first calculate the inverse function.*

Example 3.5.10. If $f(x) = \sin(x) + 3x + 2$, find $\left(f^{-1}\right)'(2)$.
By plotting $y = f(x)$, we see that f is one-to-one. Now by inspection, we see that $f(0) = 2$, so $f^{-1}(2) = 0$. We also have that $f'(x) = \cos(x) + 3$. Thus, by the inverse function theorem,

$$\begin{aligned}\left(f^{-1}\right)'(2) &= \frac{1}{f'(f^{-1}(2))} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{\cos(0) + 3} \\ &= \frac{1}{4}.\end{aligned}$$

Example 3.5.11. If $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$, find $\left(f^{-1}\right)'(6)$.
By plotting $y = f(x)$, we see that f is one-to-one. Now, by inspection, we see that $f(1) = 6$, so $f^{-1}(6) = 0$. We also have that $f'(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1$. Thus, by the inverse function theorem,

$$\begin{aligned}\left(f^{-1}\right)'(6) &= \frac{1}{f'(f^{-1}(6))} \\ &= \frac{1}{f'(1)} \\ &= \frac{1}{5(1)^4 + 4(1)^3 + 3(1)^2 + 2(1) + 1} \\ &= \frac{1}{15}.\end{aligned}$$

3.6 Derivatives of Logarithmic Functions

Theorem 3.6.1. *For $b > 0$, the function $f(x) = \log_b x$ is differentiable, and*

$$f'(x) = \frac{1}{x \ln b}.$$

Proof. Since $y = \log_b x$, then $x = b^y$. Differentiating implicitly,

$$\begin{aligned} \frac{d}{dx}[x] &= \frac{d}{dx}[b^y] \\ 1 &= b^y \ln(b) \cdot \frac{dy}{dx} \\ \Rightarrow \quad \frac{dy}{dx} = \frac{1}{b^y \ln(b)} &= \frac{1}{x \ln(b)} \end{aligned}$$

□

Corollary 3.6.2. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$.

Example 3.6.3. Compute $\dfrac{d}{dx}[\ln(x+9)]$.

By the chain rule, we have

$$\frac{d}{dx}[\ln(x+9)] = \frac{1}{x+9} \cdot \frac{d}{dx}[x+9] = \frac{1}{x+9}.$$

Example 3.6.4. Find $f'(x)$ where $f(x) = \log_2(x^2 + \cos x)$

Notice that $f(x) = g(h(x))$, where $g(x) = \log_2(x)$ and $h(x) = x^2 + \cos x$. We have that

$$g'(x) = \frac{1}{x \ln 2} \quad \text{and} \quad h'(x) = 2x - \sin x,$$

so applying the chain rule, we get

$$f'(x) = g'(h(x)) \cdot h'(x) = \frac{1}{(x^2 + \cos x) \ln 2} \cdot (2x - \sin x) = \frac{2x - \sin x}{(x^2 + \cos x) \ln 2}.$$

Example 3.6.5. Find $\frac{dg}{du}$ where $g(u) = \frac{1+u}{1+\ln u}$.

Applying the quotient rule, we have

$$\frac{dg}{du} = \frac{1(1 + \ln u) - (1+u)\frac{1}{u}}{(1 + \ln u)^2} = \frac{\ln u - \frac{1}{u}}{(1 + \ln u)^2} = \frac{u \ln u - 1}{u(1 + \ln u)^2}.$$

Example 3.6.6. Find y' for $y = \ln|x|$.

Following from the definition of the absolute value, we have

$$\ln|x| = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0. \end{cases}$$

So, if $x > 0$, we have

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}\ln x = \frac{1}{x},$$

and if $x < 0$, we have

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus $y' = \frac{1}{x}$ for all $x \neq 0$.

Example 3.6.7. Differentiate $f(x) = \ln\left(\frac{x^2+3}{x^2+1}\right)$.

Rather than going straight for the quotient rule within the chain rule, we can use the summation property of logarithms to rewrite this:

$$\begin{aligned} f(x) &= \ln\left(\frac{x^2+3}{x^2+1}\right) = \ln(x^2+3) - \ln(x^2+1) \\ \Rightarrow \quad f'(x) &= \frac{1}{x^2+3} \cdot \frac{d}{dx}\left[x^2+3\right] - \frac{1}{x^2+1} \cdot \frac{d}{dx}\left[x^2+1\right] \\ &= \frac{2x}{x^2+3} - \frac{2x}{x^2+1} \end{aligned}$$

3.6.1 Logarithmic Differentiation

The previous example shows us that logarithms can be used to simplify quotient/product rules. following example gives a procedure, known as **logarithmic differentiation**, for finding derivatives that may involve copious amounts of product/quotient rules, as well as derivatives of exponential functions (which is something we don't have yet).

Example 3.6.8. Differentiate $y = \dfrac{(x^2-7)\sqrt{x^{5/2}+x}}{(2x+9)^{750}}$.

Since logarithms are injective, we take a logarithm of both sides and use the properties of logarithms to simplify the expression:

$$\begin{aligned} \ln y &= \ln(x^2-7) + \ln\sqrt{x^{5/2}+x} - \ln(2x+9)^{750} \\ &= \ln(x^2-7) + \frac{1}{2}\ln(x^{5/2}+x) - 750\ln(2x+9). \end{aligned}$$

Differentiating implicitly with respect to x gives us

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2-7} + \frac{1}{2} \cdot \frac{\frac{5}{2}x+1}{x^{5/2}+x} - 750 \cdot \frac{2}{2x+9},$$

and solving for $\frac{dy}{dx}$, we get

$$\begin{aligned}\frac{dy}{dx} &= y\left(\frac{2x}{x^2-7} + \frac{1}{2}\cdot\frac{\frac{5}{2}x+1}{x^{5/2}+x} - 750\cdot\frac{2}{2x+9}\right) \\ &= \frac{(x^2-7)\sqrt{x^{5/2}+x}}{(2x+9)^{750}}\left(\frac{2x}{x^2-7} + \frac{1}{2}\cdot\frac{\frac{5}{2}x+1}{x^{5/2}+x} - 750\cdot\frac{2}{2x+9}\right).\end{aligned}$$

Procedure for logarithmic differentiation:

1. Take the natural logarithm of both sides of the equation $y = f(x)$.
2. Simplify with the properties of logarithms.
3. Implicitly differentiate with respect to x.
4. Solve the resulting equation for y'.

Logarithmic differentiation allows us to actually prove the power rule for all real numbers.

Proposition 3.6.9 (Power Rule)**.** *If r is any real number and $f(x) = x^r$, then*

$$f'(x) = rx^{r-1}.$$

Proof. If $x < 0$, it may be that $\ln x^r$ is undefined. Fortunately, the result of Example **??** tells us that we can use $\ln|x^r|$ instead. Recall also a property of absolute values that $|x^r| = |x|^r$. So, setting $y = x^r$, and using logarithmic differentiation, we have

$$\ln|y| = \ln|x^r| = \ln|x|^r = r\ln|x|.$$

By Example **??**, differentiating implicitly with respect to x gets us

$$\begin{aligned} \frac{y'}{y} &= \frac{r}{x} \\ \Rightarrow \quad y' &= y\frac{r}{x} = \frac{rx^r}{x} = rx^{r-1}. \end{aligned}$$

□

Example 3.6.10. Differentiate $f(x) = x^x$.

Using logarithmic differentiation

$$\begin{aligned} \ln(f(x)) &= \ln(x^x) = x\ln(x) \\ \frac{f'(x)}{f(x)} &= \ln(x) + 1 \\ f'(x) &= f(x)\ln(x) + f(x) \\ &= x^x\ln(x) + x^x \end{aligned}$$

Example 3.6.11. Differeniate $y = [\cos(x)]^{e^x}$

Using logarithmic differentiation,

$$\begin{aligned} \ln(y) &= \ln\left([\cos(x)]^{e^x}\right) = e^x \ln(\cos(x)) \\ \Rightarrow \quad \frac{y'}{y} &= e^x \ln(\cos(x)) + e^x \cdot \frac{-\sin(x)}{\cos(x)} \\ y' &= \frac{e^x \ln(\cos(x)) - \frac{e^x \sin(x)}{\cos(x)}}{[\cos(x)]^{e^x}}. \end{aligned}$$

Example 3.6.12. Would you benefit from using logarithmic differentiation to find $\frac{dy}{d\theta}$ for the equation $y = \theta \cos \theta$?

No. It's just a quick product rule.

Example 3.6.13. Would you benefit from using logarithmic differentiation to find $\frac{dy}{d\theta}$ for the equation $y = \theta \cos^2 \theta$?

Probably not. It's just a quick product rule and a chain rule.

Example 3.6.14. Would you benefit from using logarithmic differentiation to find $\frac{dy}{d\theta}$ for the equation $y = \left[\frac{\theta \cos^2 \theta}{\sqrt{\sec \theta}}\right]^\theta$?

Okay, yes.

First notice that we can simplify this a little bit. Because $\sqrt{\sec \theta} = 1/\sqrt{\cos \theta}$, it follows that

$$y = \left[\theta \cos^{5/2} \theta\right]^\theta.$$

Using logarithmic differentiation

$$\begin{aligned} \ln(y) &= \ln\left[(\theta \cos^{5/2} \theta)^\theta\right] = \theta \left[\ln(\theta) + \frac{5}{2} \ln(\cos(\theta))\right] \\ \Rightarrow \quad \frac{1}{y}\frac{dy}{d\theta} &= \left[\ln(\theta) + \frac{5}{2} \ln(\cos(\theta))\right] + \theta \left[\frac{1}{\theta} + \frac{5}{2} \cdot \frac{-\sin \theta}{\cos \theta}\right] \\ \frac{dy}{d\theta} &= (\theta \cos^{5/2} \theta)^\theta \cdot \left[\ln(\theta) + \frac{5}{2} \ln(\cos(\theta))\right] + \theta \left[\frac{1}{\theta} + \frac{5}{2} \cdot \frac{-\sin \theta}{\cos \theta}\right]. \end{aligned}$$

3.6.2 The number e as a Limit

Let $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. So

$$1 = f'(1) = \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} = \lim_{h\to 0} \frac{1}{h}\ln(1+h) = \lim_{h\to 0} \ln\big([1+h]^{1/h}\big)$$

Since e^x is continuous, then by Theorem 2.5.9

$$\begin{aligned} 1 &= \lim_{h\to 0} \ln\big([1+h]^{1/h}\big) \\ e^1 &= e^{\lim_{h\to 0} \ln\left([1+h]^{1/h}\right)} \\ &= \lim_{h\to 0} [1+h]^{1/h}. \end{aligned}$$

From this limit we have

Definition. $e^x = \lim_{h\to 0} (1+h)^{x/h}$

alternatively, making the substitution $n = \frac{x}{h}$ and noting that $n \to \infty$ as $h \to 0$, we get

Definition. $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$

3.7 Rates of Change in the Natural Sciences

Suppose we have an object moving along a straight line. Let $s = s(t)$ be a function representing the object's position along this line at time t.

Definition. The **displacement** of the object from time t_1 to t_2 is $s(t_2) - s(t_1)$.

Remark. Displacement contains information about distance and direction. If we claim that "forward" is the positive direction and "backward" is the negative direction, then a negative displacement D corresponds to moving backwards a distance of $|D|$.

Suppose an object moves forward on the time interval $[t_1, t_2]$, then backward on the time interval $[t_2, t_3]$, etc.

Definition. The **total distance traveled** by the object from time t_1 to t_n is

$$|s(t_n) - s(t_{n-1})| + |s(t_{n-1}) - s(t_{n-2})| + \cdots + |s(t_2) - s(t_1)|$$

where each t_i corresponds to a change in direction of travel.

Definition. The **average velocity** from time t to time t_1 is $\dfrac{s(t_1) - s(t)}{t_1 - t}$.

Definition. The **(instantaneous) velocity** at time t is $v(t) = \lim\limits_{t_1 \to t} \dfrac{s(t_1) - s(t)}{t_1 - t} = s'(t)$.

Remark. Velocity contains information about speed *and* direction. If we claim that "forward" is the positive direction and "backward" is the negative direction, then a negative velocity v corresponds to moving backwards at a speed of $|v|$.

Definition. The **(instantaneous) acceleration** at time t is $a(t) = v'(t)$.

Remark. For positive velocity, when acceleration is positive, the object's speed is increasing. When acceleration is negative, the object's speed is decreasing.
For negative velocity, when acceleration is positive, the object's speed is decreasing. When acceleration is positive, the object's speed is increasing.

Definition. The object is **speeding up** if $v(t), a(t) > 0$ or if $v(t), a(t) < 0$ (i.e. velocity and acceleration have the same sign). The object is **slowing down** if $v(t) < 0 < a(t)$ or $v(t) > 0 > a(t)$ (i.e. the velocity and acceleration have opposite signs).

Example 3.7.1. A particle is moving along a straight line. Its position (measured in feet from its starting position) at time t is given by

$$s(t) = t^3 - 6t^2 + 9t$$

a. Find the velocity at time t.

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

b. What is the velocity after 2 s? After 4 s?

$$v(2) = 12 - 24 + 9 = -3\,\text{ft/s} \qquad v(4) = 48 - 24 + 9 = 33\,\text{ft/s}$$

c. When is the particle at rest?

$$0 = 3t^2 - 12t + 9 = 3(t-1)(t-3) \qquad \Longrightarrow \qquad t = 1\,\text{s and } t = 3\,\text{s}$$

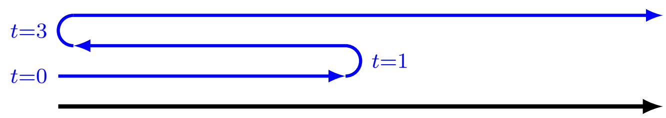
d. When is the particle moving forward (that is, in the positive direction)?

From the part (c), the velocity is positive when $t < 1\,\text{s}$ and $t > 3\,\text{s}$.

e. When is the particle moving backward (that is, in the negative direction)?

From the part (c), the velocity is negative when $1\,\text{s} < t < 3\,\text{s}$.

f. Draw a diagram to represent the motion of the particle.



g. Find the total distance traveled by the particle during the first five seconds.

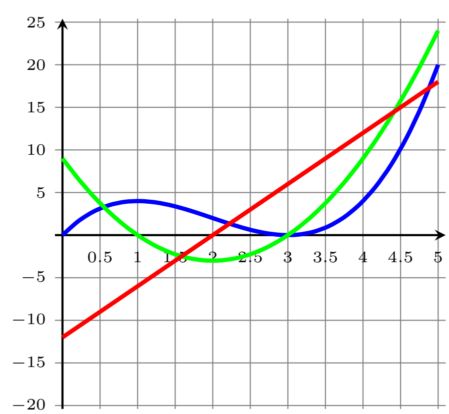
We have to account for where the object backtracks as well, so we compute the distance traveled on each of the following time intervals: $[0\,\text{s}, 1\,\text{s}]$, $[1\,\text{s}, 3\,\text{s}]$, and $[3\,\text{s}, 5\,\text{s}]$:

$$\text{distance} = |s(1) - s(0)| + |s(3) - s(1)| + |s(5) - s(3)| = |4 - 0| + |0 - 4| + |20 - 0| = 28\,\text{ft}.$$

h. Find the acceleration at time t and after $4\,\text{s}$.

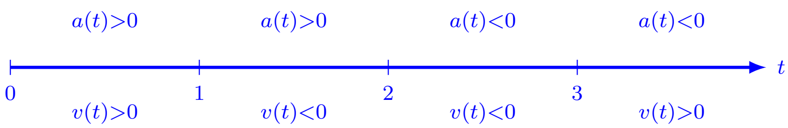
$$a(t) = v'(t) = 6t - 12$$

i. Graph the position ($y = s(t)$), velocity ($y = v(t)$), and acceleration ($y = a(t)$) functions for $0 \le t \le 5$.



j. When is the particle speeding up? When is it slowing down?

$a(t)$ is positive when $t > 2\,\text{s}$ and $a(t)$ is negative when $t < 2\,\text{s}$. Comparing these signs with the positive/negative velocity from earlier:



Hence the object is speeding up on the intervals $(0, 1)$ and $(2, 3)$ and it is slowing down on the intervals $(1, 2)$ and $(4, \infty)$.

Example 3.7.2. A particle is moving along a straight line. Its position from the origin (measured in meters) at time t is given by

$$s(t) = t^2 e^{-t}$$

a. Find the velocity at time t.

$$v(t) = s'(t) = 2te^{-t} - t^2e^{-t} \qquad \text{(product rule)}$$

b. What is the velocity after 1 s?

$$v(1) = 2(1)e^{-1} - 1e^{-1} = \frac{1}{e}\,\text{m/s}$$

c. When is the particle at rest?

$$0 = v(t) = t(2-t)e^{-t} \qquad \Longrightarrow \qquad t = 0\,\text{s and } t = 2\,\text{s}.$$

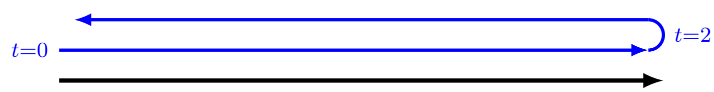
d. When is the particle moving forward (that is, in the positive direction)?

From part (c), $v(t)$ is positive when $0\,\text{s} < t < 2\,\text{s}$.

e. When is the particle moving backward (that is, in the negative direction)?

From part (c), $v(t)$ is negative when $t > 2\,\text{s}$ (since we're modeling the physical world, we ignore negative time).

f. Draw a diagram to represent the motion of the particle.



g. Find the total distance traveled by the particle during the first 6 seconds.

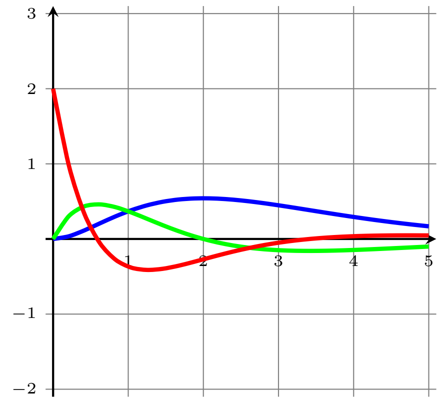
$$\text{distance} = |s(2) - s(0)| + |s(6) - s(2)| = 8e^{-2} - 36e^{-6}\,\text{m} \approx 0.99\,\text{m}$$

h. Find the acceleration at time t and after 1 s.

$$a(t) = v'(t) = 2e^{-t} - 2te^{-t} - (2te^{-t} - t^2e^{-t}) = e^{-t}(t^2 - 4t + 2)$$

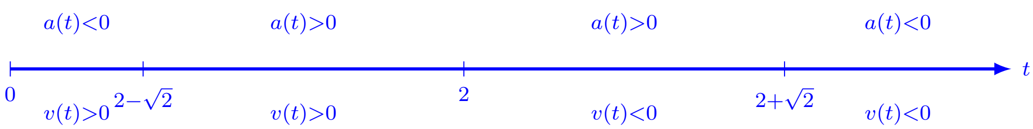
$$a(1) = -\frac{1}{e}\,\text{m/s}^2 \approx -0.37\,\text{m/s}^2.$$

i. Graph the position ($y = s(t)$), velocity ($y = v(t)$), and acceleration ($y = a(t)$) functions for $0 \leq t \leq 5$.



j. When is the particle speeding up? When is it slowing down?

Since e^{-t} is always positive, $a(t)$ is positive when $2 - \sqrt{2}\,\text{s} < t < 2 + \sqrt{2}\,\text{s}$; $a(t)$ is negative when $t < 2 - \sqrt{2}\,\text{s}$ and $t > 2 + \sqrt{2}\,\text{s}$. Comparing this with the velocities we computed in a previous part:



Hence the object is speeding up on the intervals $(2 - \sqrt{2}, 2)$ and $(2 + \sqrt{2}, \infty)$ and it is slowing down on the intervals $(0, 2 - \sqrt{2})$ and $(2, 2 + \sqrt{2})$.

Example 3.7.3. A mass is hanging from a spring in equilibrium 15 cm above the ground. At time $t = 0$, the mass is pulled 9 cm down from its equilibrium position and released. The height of the mass at time t is given by

$$s(t) = -9e^{-\lambda t/2} \cos\left(\frac{\pi}{4} t\right) + 15$$

for some positive real number λ (the coefficient of drag force). In what follows, let $\lambda = \frac{\pi}{2}$ (this is really unreasonable for Earth's atmosphere, but computationally nicer).

a. Find the velocity at time t.

$$v(t) = s'(t) = \frac{9\pi}{4} e^{-\pi t/4} \left(\cos\left(\frac{\pi}{4} t\right) + \sin\left(\frac{\pi}{4} t\right)\right)$$

b. What is the velocity after 2 s?

$$v(2) = \frac{9\pi}{4} e^{-\pi/4}$$

c. When is the mass at rest in the time interval $(0\,\text{s}, 9\,\text{s})$?

$$\begin{aligned} 0 = v(t) &= \frac{9}{4}e^{-\pi t/4}\left(\cos\left(\frac{\pi}{4}t\right) + \sin\left(\frac{\pi}{4}t\right)\right) \\ \implies \quad & \cos\left(\frac{\pi}{4}t\right) = -\sin\left(\frac{\pi}{4}t\right) \\ \implies \quad & \frac{\pi}{4}t = 3\pi/4 + n\pi \\ \implies \quad & t = 3\,\text{s}, 7\,\text{s} \end{aligned}$$

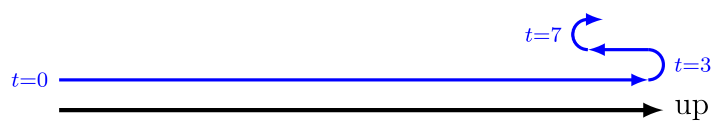
d. When is the mass moving upward (that is, in the positive direction) during the time interval $(0\,\text{s}, 9\,\text{s})$?

Testing points in the intervals $(0, 3)$, $(3, 7)$, and $(7, 9)$, we see that $v(t) > 0$ when $0\,\text{s} < t < 3\,\text{s}$ and $7\,\text{s} < t < 9\,\text{s}$.

e. When is the mass moving downward (that is, in the negative direction) during the time interval $(0\,\text{s}, 9\,\text{s})$?

Testing points in the intervals $(0, 3)$, $(3, 7)$, and $(7, 9)$, we see that $v(t) < 0$ when $3\,\text{s} < t < 7\,\text{s}$.

f. Draw a diagram to represent the motion of the particle during the time interval $(0\,\text{s}, 9\,\text{s})$.



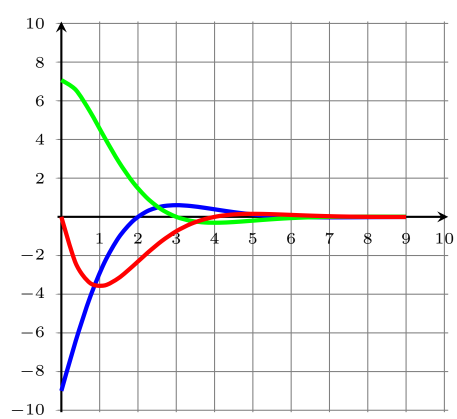
g. Find the total distance traveled by the particle during the first 9 seconds.

$$\text{distance} = |s(3) - s(0)| + |s(7) - s(3)| + |s(9) - s(7)| \approx 8.40 + 0.63 + 0.02\,\text{cm} = 9.05\,\text{cm}.$$

h. Find the acceleration at time t and after $1\,\text{s}$.

$$a(t) = v'(t) = -\frac{9\pi^2}{8}e^{-\pi t/4}\sin\left(\frac{\pi}{4}t\right).$$

i. Graph the position ($y = s(t)$), velocity ($y = v(t)$), and acceleration ($y = a(t)$) functions for $0 \le t \le 9$.



3.9 Related Rates

We will motivate this topic with the following example.

Example 3.9.1. Dr. Wells is blowing up a balloon using regular, consistent breaths (exhaling roughly 1 L of air each time). The balloon is nearly spherical, so he records the radius of the balloon after each breath:

balloon volume (in L $= 1000\text{cm}^3$)	0.0	1.0	2.0	3.0	4.0
balloon radius (in cm)	0.0	6.2	7.7	9.0	9.8

The volume and radius are both increasing with time. However, while the volume is increasing at a constant rate $\frac{dV}{dt}$, the rate that the radius is increasing, $\frac{dr}{dt}$, is not constant. So how are $\frac{dV}{dt}$ and $\frac{dr}{dt}$ related?

Recall that for a sphere, $V = \frac{4}{3}\pi r^3$. Since the radius $r = r(t)$ and the volume $V = V(t)$ are both functions of time, we can implicitly differentiate both sides of this equation (with respect to time), and that will give us an equation relating $\frac{dV}{dt}$ and $\frac{dr}{dt}$:

$$\begin{aligned} \frac{d}{dt}[V] &= \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] \\ \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt}. \end{aligned}$$

Example 3.9.2. A stone is dropped into a calm lake, which causes concentric circular ripples to emanate from the splash point. The radius r of the outermost ripple is increasing a rate of 2 feet per second. When the radius gets to be 7 feet, at what rate is the total area A of the rippled water changing?

Recall that $A = \pi r^2$. Using implicit differentiation, we see that the changing area is related to the changing radius by

$$\begin{aligned} \frac{d}{dt}[A] &= \frac{d}{dt}[\pi r^2] \\ \frac{dA}{dt} &= 2\pi r \frac{dr}{dt}. \end{aligned}$$

We're given that $\frac{dr}{dt} = 2\,\text{ft/s}$, so when $r = 7\,\text{ft}$, we have

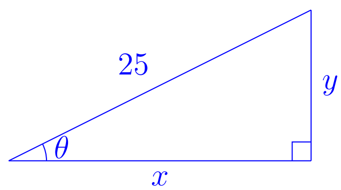
$$\frac{dA}{dt} = 2\pi(7)(2) = 28\pi\ \text{ft}^2/\text{s}.$$

Example 3.9.3. A 25-foot ladder is leaning against the wall of a building. The base of the ladder is being pulled away from the building at a rate of 2 feet per second, and the top of the ladder is sliding down the wall.

a. How fast is the top of the ladder sliding down the wall when the base is 8 feet away from the wall?

b. At what rate is the angle between the ladder and the ground changing when the base is 8 feet away from the wall?

Let $x = x(t)$ represent the distance of the base of the ladder from the wall, $y = y(t)$ be the height of the top of the latter, and $\theta = \theta(t)$ the angle formed between the ground and the base of the ladder.



a. We want to relate $\frac{dx}{dt}$ and $\frac{dy}{dt}$. From the picture above, it's clear that x and y are related by

$$x^2 + y^2 = 25^2 = 625.$$

With implicit differentiation, we have that

$$\begin{aligned} \frac{d}{dt}\left[x^2 + y^2\right] &= \frac{d}{dt}[625] \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ \Rightarrow \quad \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt} = -\frac{x}{\sqrt{625 - x^2}}\frac{dx}{dt}. \end{aligned}$$

We're given that $\frac{dx}{dt} = 2\,\text{ft/s}$, so when $x = 8$, we have

$$\frac{dy}{dt} = -\frac{2(8)}{2\sqrt{625 - 64}}(2) = -\frac{16}{\sqrt{561}} \approx -0.676\,\text{ft/s}.$$

Thus the top of the ladder is sliding down at a rate of roughly 0.676 ft/s.

b. Now we want to relate $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$. Again, from the picture above, we have that x, y, and θ are related by $\cos\theta = \frac{x}{25}$ and $\sin\theta = \frac{y}{25}$. With implicit differentiation,

$$\begin{aligned} \frac{d}{dt}[\cos\theta] &= \frac{d}{dt}\left[\frac{x}{25}\right] \\ -\sin\theta\frac{d\theta}{dt} &= \frac{1}{25}\frac{dx}{dt} \\ \Rightarrow \quad \frac{d\theta}{dt} &= -\frac{1}{25\sin\theta}\frac{dx}{dt} = -\frac{1}{y}\frac{dx}{dt} = -\frac{1}{\sqrt{625 - x^2}}\frac{dx}{dt}. \end{aligned}$$

We're given that $\frac{dx}{dt} = 2\,\text{ft/s}$, so when $x = 8$, we get

$$\frac{d\theta}{dt} = -\frac{1}{\sqrt{625 - 64}}(2) = -\frac{2}{\sqrt{561}} \approx -0.084\,\text{rad/s} \approx -4.838\,\text{deg/s}.$$

Example 3.9.4. A perfectly spherical balloon is being filled with air at a constant rate of 10 cubic inches per minute. At some point in time, an observer measures that the radius is increasing at a rate of 1.7 inches per minute. What is the radius of the balloon when this measurement is taken, and what is the volume of the balloon when this measurement is taken?

Let $r = r(t)$ be the radius of the balloon and $V = V(t)$ the volume of the balloon at time t. Recall that the volume of a sphere is given by

$$V = \frac{4}{3}\pi r^3.$$

So, differentiating both sides of this equation with respect to t, we have

$$\begin{aligned} \frac{d}{dt}[V] &= \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] \\ \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt}. \end{aligned} \tag{3.9.1}$$

We're given that $\frac{dV}{dt} = 10\,\text{in}^3/\text{min}$ and $\frac{dr}{dt} = 1.7\,\text{in}/\text{min}$, so rearranging Equation 3.9.1 to solve for r, we get that

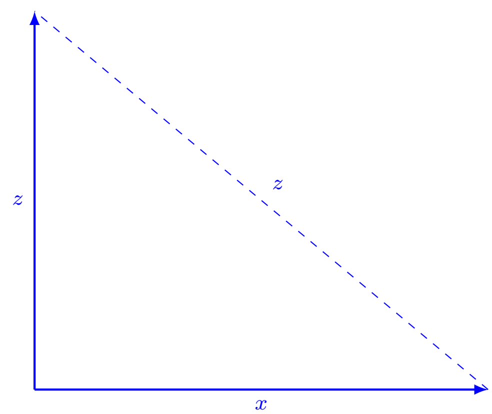
$$\begin{aligned} r &= \sqrt{\frac{\frac{dV}{dt}}{4\pi\frac{dr}{dt}}} \\ &= \sqrt{\frac{10}{4\pi(1.7)}} \\ &\approx 0.684\,\text{in}, \end{aligned}$$

at the time the measurement is taken. The equation for the volume of the sphere tells us that the balloon's volume is

$$V \approx \frac{4}{3}\pi(0.684) \approx 1.34\,\text{in}^3$$

at the time of the measurement.

Example 3.9.5. A plane is flying due north at a constant 500 kilometers per hour and another is flying due east at a constant 600 kilometers per hour. If both planes pass each other at some time, at what rate is the horizontal distance between the planes changing two hours later?[†]



Let $y = y(t)$ be the distance traveled north and $x = x(t)$ be the distance traveled east after t hours have passed. By the Pythagorean Theorem, we have that the distance $z = z(t)$ between the two planes after t hours is given by

$$z^2 = x^2 + y^2.$$

Differentiating both sides with respect to t, we have

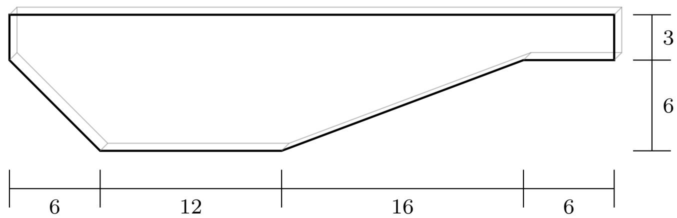
$$\begin{aligned} \frac{d}{dt}\left[z^2\right] &= \frac{d}{dt}\left[x^2 + y^2\right] \\ 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \Rightarrow \quad \frac{dz}{dt} &= \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt} \\ &= \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}. \end{aligned} \tag{3.9.2}$$

After 2 hours, we have that $x = 1200\,\text{km}$ and $y = 1000\,\text{km}$. Since we're given that $\frac{dx}{dt} = 600\,\text{km/h}$ and $\frac{dy}{dt} = 500\,\text{km/h}$, we plug all of this into Equation 3.9.2 to get

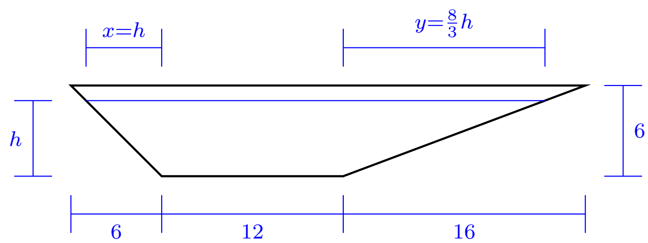
$$\frac{dz}{dt} = \frac{1200}{\sqrt{1200^2 + 1000^2}}(600) + \frac{1000}{\sqrt{1200^2 + 1000^2}}(500) \approx 781\,\text{km/h}.$$

[†] Many thanks to Sanmeel Lagad for pointing out that a previous version of these notes ommited a crucial assumption in the problem statement.

Example 3.9.6. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure below. If the pool is being filled at a rate of $0.8\,\text{ft}^3/\text{min}$, how fast is the water level rising when the depth at the deepest point is 5 ft?



Notice that the rate of change of the volume will be different when the water level is above 6 ft and when it is below 6 ft. As such, we are in the latter case. Let $h = h(t)$ be the height of the water.



(Here x and y were determined by similar triangles). We thus have that the volume $V = V(t)$ of the pool at height h is given by

$$V = 20\left[\frac{1}{2}(h)h + \frac{1}{2}\left(\frac{8}{3}h\right)h + 12h\right] = \frac{110}{3}h^2 + 240h.$$

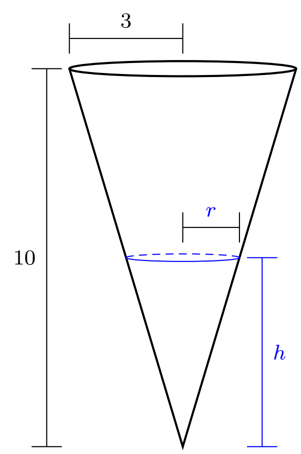
Differentiating both sides with respect to t yields

$$\begin{aligned} \frac{d}{dt}[V] &= \frac{d}{dt}\left[\frac{110}{3}h^2 + 240h\right] \\ \frac{dV}{dt} &= \frac{220}{3}h\frac{dh}{dt} + 240\frac{dh}{dt} \\ &= \left[\frac{220}{3}h + 240\right]\frac{dh}{dt} \\ \Rightarrow \quad \frac{dh}{dt} &= \frac{1}{\frac{220}{3}h + 240}\frac{dV}{dt}. \end{aligned}$$

We're given that $\frac{dV}{dt} = 0.8\,\text{ft}^3/\text{min}$, so when the water level is 5 ft, we get

$$\frac{dh}{dt} = \frac{1}{\frac{220}{3}(5) + 240}(0.8) \approx 0.00132\,\text{ft}/\min.$$

Example 3.9.7. A conical tank with height 10 m and a top radius of 3 m is filled with water. The tank is being drained at the rate of $0.5\,\mathrm{m}^3/\mathrm{s}$. Given that the volume of a cone is $V = \frac{\pi}{3}r^2h$, how fast is the height of the water changing when the surface of the water has a radius of 1 m? *Hint: similar triangles*



Since the tank is being drained, we remark that $\frac{dV}{dt} = -0.5\,\mathrm{m}^3/\mathrm{s}$. Now, differentiate the equation of the volume of the cone to relate the rates.

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{3}r^2h\right]$$
$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) \qquad \text{(product rule).}$$

So now we know how the rates $\frac{dV}{dt}$, $\frac{dr}{dt}$, and $\frac{dh}{dt}$ are related. We're not told anything about the value of h or $\frac{dr}{dt}$, but we can figure out each of these from similar triangles.

$$\frac{h}{10} = \frac{r}{3}, \qquad \text{hence} \qquad \frac{1}{10}\frac{dh}{dt} = \frac{1}{3}\frac{dr}{dt}.$$

From these equations, we can rewrite our related rates equation above as

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2r\left(\frac{10r}{3}\right)\left(\frac{3}{10}\frac{dh}{dt}\right) + r^2\frac{dh}{dt}\right)$$
$$= \frac{\pi}{3}\left(3r^2\right)\frac{dh}{dt}$$

Substituting in our given information of $r = 1$ and $\frac{dV}{dt} = 0.5$, then solving for $\frac{dh}{dt}$, we get

$$-0.5 = \frac{\pi}{3}(3)\frac{dh}{dt}$$
$$\Rightarrow \qquad \frac{dh}{dt} = \frac{-0.5}{\pi}\,\mathrm{m/s} \approx -0.159\,\mathrm{m/s}.$$

The height is decreasing at a rate of $0.159\,\mathrm{m/s}$.

Section 3.9 Exercises

Exercise 3.9.1. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

Exercise 3.9.2.

a. If A is the area of a circle with radius r and the circle expands as time passes, find $\frac{dA}{dt}$ in terms of $\frac{dr}{dt}$.

b. During an exhibition, artist Damien Hirst pours red paint onto the ground and it spreads in a circular fashion. If the radius of this paint is increasing at a constant rate of 8 in/s, how fast is the area increasing when the radius is 29 in?

Exercise 3.9.3. A rectangle of fixed perimeter is has length ℓ increasing at 5 cm/s and the width w is decreasing at 5 cm/s. How fast is the area increasing when the length is 20 cm centimeters and the width is 30 cm? When the length is 40 cm and the width is 10 cm, is the area increasing or decreasing?

Exercise 3.9.4. A perfectly spherical balloon is being inflated so that the radius is increasing at a rate of 3 mm/s. How fast is the volume increasing when the radius is 75 mm?

Exercise 3.9.5. At 12:00PM a FedEx truck is 100 miles east of a UPS truck. The FedEx truck is driving west at a 55 mi/h and the UPS truck is driving north at 75 mi/h. How fast is the distance between these trucks changing at 2:00PM?

Exercise 3.9.6. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-coordinate is decreasing at a rate of 3 units/s. How is the x-coordinate changing at that time?

Exercise 3.9.7. A kite 100 m above the ground moves horizontally at a speed of 8 m/2. At what rate is the angle between the string and the horizontal decreasing when 200 m of string has been let out?

3.10 Linear Approximation and Differentials

As we have seen, given a function f and a point a in the domain, the tangent line at the point $(a, f(a))$ is a fairly accurate approximation of the function values near a. At the point $(x_0, y_0) = (a, f(a))$, the equation of the tangent line is given by

$$\begin{aligned}(y - y_0) &= m(x - x_0)\\ (y - f(a)) &= f'(a)(x - a)\\ \Rightarrow \quad y &= f(a) + f'(a)(x - a).\end{aligned}$$

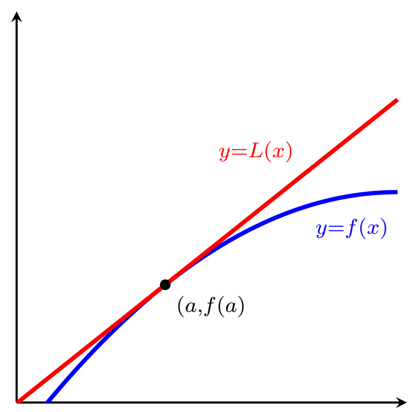
Definition. For all x near a point a, the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.



Why do we care about linear approximations? Functions can be very computationally complex (just consider how long it takes to do long division...way too long!), but linear functions are incredibly simple to compute with. So, we can approximate values of a rather complicated function with simpler linear functions to any degree of accuracy we need.

Example 3.10.1. Use the tangent line approximation of

$$f(x) = 1 + \sin(x)$$

at the point $(0, 1)$ to approximate the value of $f(0.1)$. How does your approximation compare to the calculator's given value of $f(0.1)$?

We're doing the tangent line approximation at $(0, 1)$, so $a = 0$. Hence

$$f(x) \approx f(0) + f'(0)(x - 0).$$

A quick computation shows that $f'(x) = \cos(x)$, so $f'(0) = 1$. Hence

$$f(x) \approx 1 + 1(x - 0) = 1 + x.$$

hence

$$f(0.1) \approx 1 + 0.1 = 1.1.$$

Indeed, according to our calculator, $f(0.1) = 1.099833\ldots$, so our approximation is very close.

Example 3.10.2. Use the linearization to approximate $\sqrt{16.5}$. We know $\sqrt{16}$, so we'll pick the function $f(x) = \sqrt{x}$ and look at the linear approximation of f at 16, so $a = 16$. The linearization is given by

$$L(x) = f(a) + f'(a)(x - a).$$

A quick computation shows that $f'(x) = \frac{1}{2}x^{-1/2}$, so $f'(16) = \frac{1}{8}$, hence

$$L(x) = 4 + \frac{1}{8}(x - 16).$$

So, $f(16.5) \approx L(16.5)$, whence we compute

$$L(16.5) = 4 + \frac{1}{8}(16.5 - 16) = 4.0625.$$

Indeed, according to our calculator, $f(16.5) = 4.0620192\ldots$, so our approximation is very close.

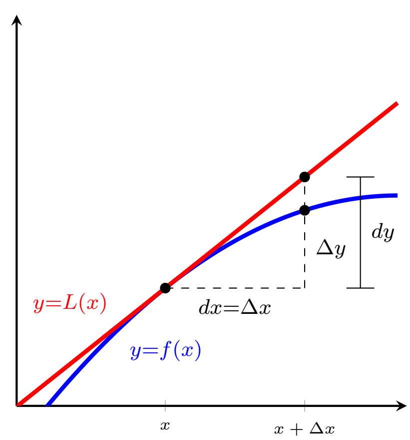
While linear approximation gives us a good estimate of our function value, sometimes we may be more concerned with how far off our approximation is. So for this we can use the language of *differentials*.

Definition. If $y = f(x)$, where f is a differentiable function, then the **differential**, dx, is an independent variable and can take on any real number, and the **differential**, dy is defined by

$$dy = f'(x)dx$$

The idea here is that small changes in the input of the function (represented by dx) can have large changes in the output (dy), and those output changes are related to the input changes by the derivative.

Geometrically, we have the following idea: if we change our function input by some small amount Δx, then the function output changes by Δy. If we change our function intput by some small amount dx, then the tangent line output changes by dy. If we pick $dx = \Delta x$ to be very small, then $dy \approx \Delta y$, so differentials do a good job of approximating the output change.



Remark. If $dx \neq 0$, we can divide both sides to get $\frac{dy}{dx} = f'(x)$, so it turns out this Leibniz notation $\frac{dy}{dx}$ is not just random notation, but rather suggests something about the slope of a function at some infinitesimal distance dx away from the point x.

Example 3.10.3. Find the differential dy given $y = f(x) = \sqrt{x^2+1}$.
We first take the derivative of f.

$$f'(x) = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$$

Since $dy = f'(x)\,dx$, we have

$$dy = f'(x)\,dx = \frac{x}{\sqrt{x^2+1}}\,dx.$$

The differentials behave in the ways you might expect.

Theorem 3.10.4 (Product Rule for Differentials). *Let f and g be differentiable functions, and let $y = f(x)g(x)$. Then*

$$dy = g(x)\,df + f(x)\,dg.$$

Proof. By definition, $df = f'(x)\,dx$ and $dg = g'(x)\,dx$. So,

$$\begin{aligned} dy &= y'\,dx \\ &= [f'(x)g(x) + f(x)g'(x)]\,dx \\ &= g(x)f'(x)\,dx + f(x)g'(x)\,dx \\ &= g(x)\,df + f(x)\,dg. \end{aligned}$$

□

Theorem 3.10.5 (Quotient Rules for Differentials). *Let f and g be differentiable functions, and let $y = \frac{f(x)}{g(x)}$. Then*

$$dy = \frac{g(x)\,df - f(x)\,dg}{[g(x)]^2}.$$

Proof. By definition, $df = f'(x)\,dx$ and $dg = g'(x)\,dx$. So,

$$\begin{aligned} dy &= y'\,dx \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}\,dx \\ &= \frac{g(x)f'(x)\,dx - f(x)g'(x)\,dx}{[g(x)]^2} \\ &= \frac{g(x)\,df - f(x)\,dg}{[g(x)]^2}. \end{aligned}$$

□

Example 3.10.6. Compute the differential dw given $w = x^{15}\cos(2x)$.
Setting $f(x) = x^{15}$ and $g(x) = \cos(2x)$, we have $w = f(x)g(x)$. A quick calculation of the differentials of f and g shows that

$$\begin{aligned} df &= 15x^{14}\,dx \\ dg &= -2\sin(2x)\,dx. \end{aligned}$$

So, by the product rule above,

$$\begin{aligned} dw &= g(x)\,df + f(x)\,dg \\ &= 15x^{14}\cos(2x)\,dx - 2x^{15}\sin(2x)\,dx \\ &= \left[15x^{14}\cos(2x) - 2x^{15}\sin(2x)\right]\,dx. \end{aligned}$$

These differentials will be immediately useful to us in error analysis.

Definition. Given a quantity Q and a measured error ΔQ, the **relative error** is given by $\dfrac{\Delta Q}{Q}$. If this fraction is expressed as a percentage, we call it the **percentage error**.

Example 3.10.7. The radius of a ball bearing is measured to be 0.7 inch. If the largest possible error in measurement is 0.01 inch, estimate the largest possible relative error and percentage error in the volume V of the bearing.

Recall that the volume of a sphere is given by

$$V = \frac{4}{3}\pi r^3,$$

where r is the radius of the bearing. If dr represents the error in our radius measurement, then the corresponding error in the calcuated value of V is approximated by the differential

$$dV = 4\pi r^2\,dr$$

When $r = 0.7\,\text{in}$ and $dr = 0.01\,\text{in}$, the our measurement in volume is off by (at most)

$$\begin{aligned} dV &= 4\pi r^2\,dr \\ &= 4\pi(0.7)^2(0.01) \\ &= 0.0616\,\text{in}^3. \end{aligned}$$

The maximum percentage error is thus given by

$$\begin{aligned} \frac{\Delta V}{V} \approx \frac{dV}{V} &= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} \\ &= \frac{3\,dr}{r} \\ &= \frac{3(0.01)}{0.7} \\ &= 0.0429 \;\text{ or }\; 4.29\%. \end{aligned}$$

Example 3.10.8. The measurements of the base and altitude of a triangle are 36 centimeters and 50 centimeters, respectively. The possible error in measurement is 0.25 cm. Use differentials to approximate the relative error in computing the area of the triangle.

Recall that the area of a triangle is given by

$$A = \frac{1}{2}bh,$$

where b is the base measurement and h is the height/altitude. The two measurements, b and h, are independent of one another, as are db and dh (respectively). Thus the differentials will satisfy the product rule, hence

$$dA = \frac{1}{2}h\,db + \frac{1}{2}b\,dh.$$

At worst, our measurements were off by a full $0.25\,cm$, so we have $db = dh = 0.25\,\text{cm}$. Thus, our maximum possible error in measurement of the area is

$$\begin{aligned} dA &= \frac{1}{2}h\,db + \frac{1}{2}b\,dh \\ &= \frac{1}{2}(36)(0.25) + \frac{1}{2}(50)(0.25) \\ &= 10.75\,\text{cm}^2. \end{aligned}$$

The percentage error is thus given by

$$\begin{aligned} \frac{\Delta A}{A} \approx \frac{dA}{A} &= \frac{\frac{1}{2}h\,db + \frac{1}{2}b\,dh}{\frac{1}{2}bh} \\ &= \frac{db}{b} + \frac{dh}{h} \\ &= \frac{0.25}{36} + \frac{0.25}{50} \\ &\approx 0.0119 \ \text{ or } \ 1.19\%. \end{aligned}$$

4 Applications of Differentiation

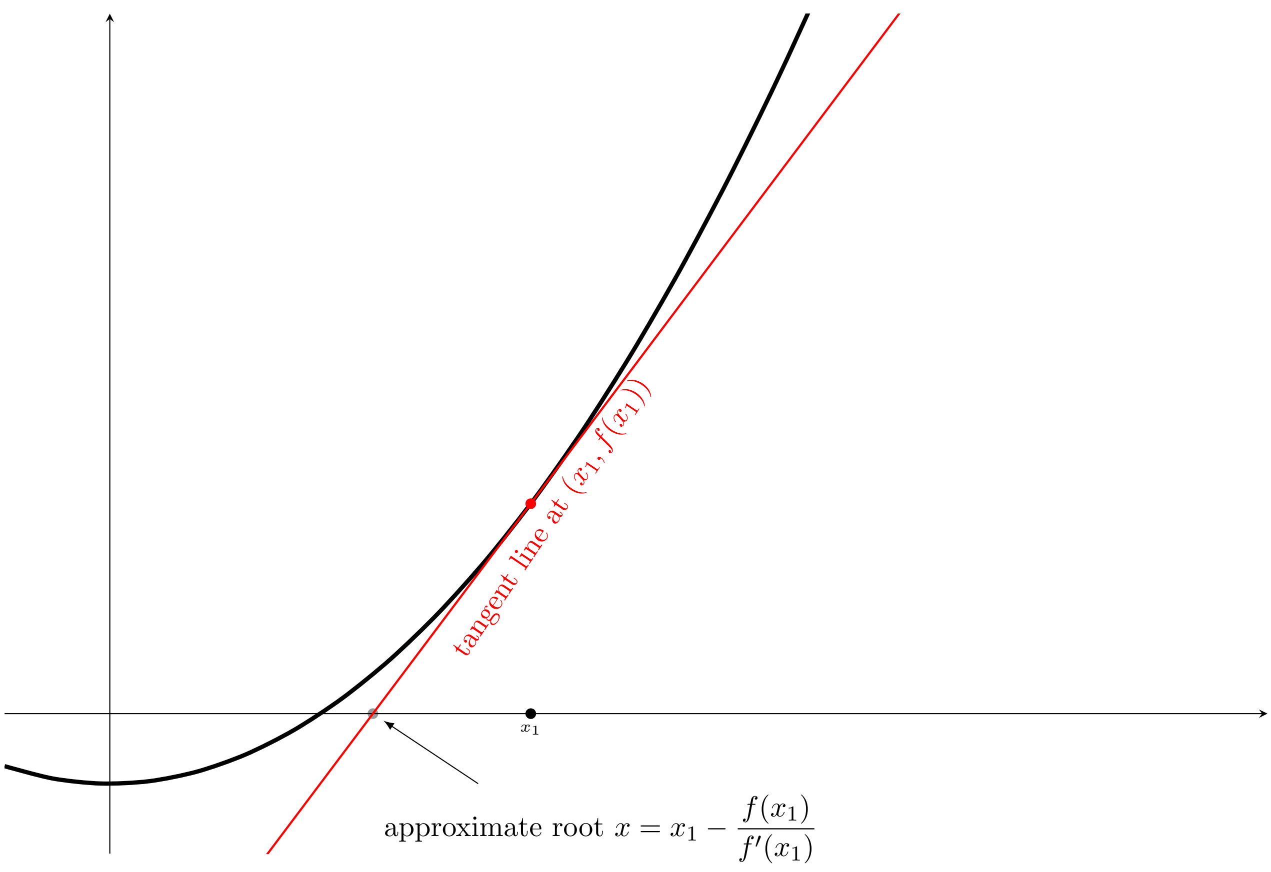
4.8 Newton's Method

Often, given a function f, we want to find x-values where $f(x) = 0$. Unfortunately, this is really hard to do for most functions, even for really nice functions like polynomials. A very famous theorem due to Abel–Rufini tells us that no version of the "quadratic formula" for polynomials of degree 5 or higher. So if we can't rely on explicit techniques to find zeroes, our only other strategy is to try to approximate the values to arbitrary precision. From the last section, we learned that the tangent line does a pretty good of job of doing just that.

Suppose we pick a point x_1 that is near where we think the root will be. Then we look at where the tangent line crosses the x-axis:

$$\begin{aligned} y - f(x_1) &= f'(x)(x - x_1) && \text{(tangent line approximation)} \\ 0 - f(x_1) &= f'(x)(x - x_1) && \text{(tangent line crosses } x\text{-axis)} \\ \implies x &= x_1 - \frac{f(x_1)}{f'(x_1)} && \text{(solve for x)} \end{aligned}$$

If we picked x_1 close enough to the actual zero of the function f, then the tangent line approximation actually gives us an *even better* approximation, x of the zero. Graphically:

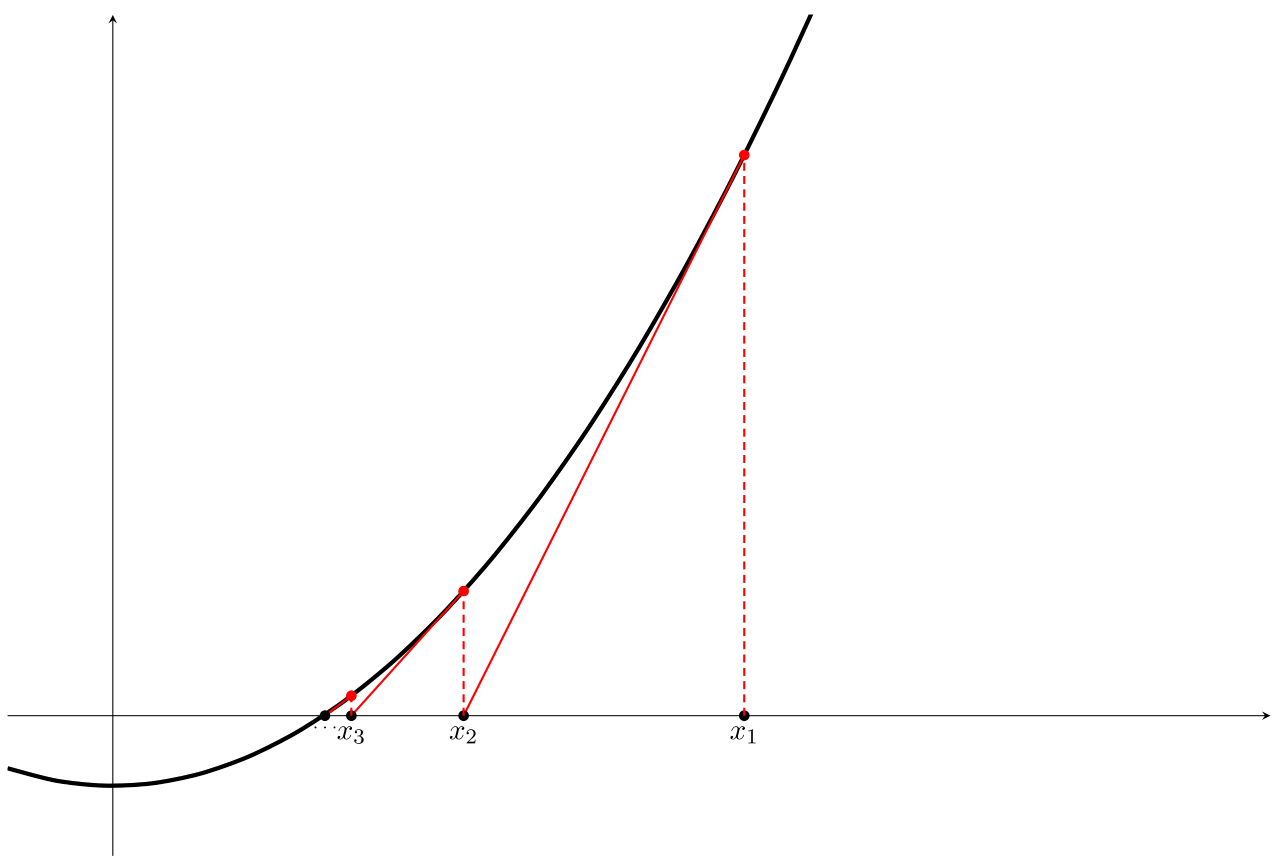


It seems reasonable, then, that one can keep doing this iteratively to find better and better approximations of the zero of the function. Indeed, this method is called **Newton's Method** or the **Newton–Rhapson Method** and is one of the most common procedures for doing so (in fact, most of your calculators use this same method).

Strategy for Newton(–Rhapson) Method

1. Guess an initial value, x_1, so that $f(x_1)$ is very close to 0.
2. Use the tangent line approximation at x_1 to determine better approximate x-value, x_2. In other words
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \qquad (f'(x_1) \neq 0)$$
3. Repeat last step using x_2 to find x_3, then x_3 to find x_3, ..., then x_n to find x_{n+1}, ... Keep repeating until you get bored or achieve the desired level of precision. The formula for the $(n+1)^{\text{th}}$ approximation is given by
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (f'(x_n) \neq 0).$$

Graphically,



See https://www.desmos.com/calculator/nvff2i4zur for an interactive Desmos graph that does the first few iterates of Newton's Method.

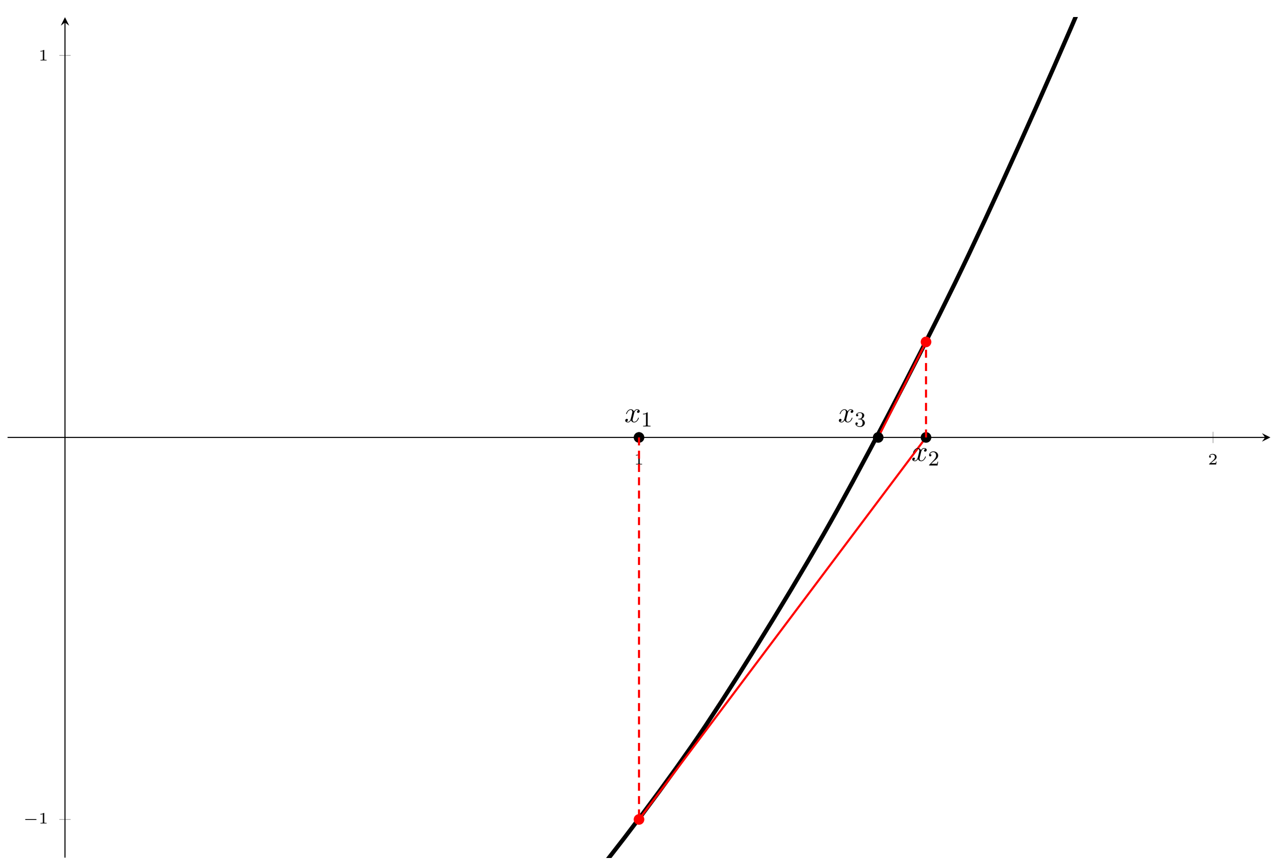
Example 4.8.1. Find $x > 0$ so that $x^2 - 2 = 0$.

We know that it should be $x = \sqrt{2} \approx 1.41421356237$), so let's see how this works with Newton's Method. Collecting our function and its derivative,

$$f(x) = x^2 - 2 \qquad \text{and} \qquad f'(x) = 2x,$$

we guess that $x_1 = 1$. We can then go about computing x_2, x_3, x_3, etc. We'll collect that all in the following table:

$\boldsymbol{n}$	$\boldsymbol{x_n}$	$\boldsymbol{f(x_n)}$	$\boldsymbol{f'(x_n)}$	$\boldsymbol{x_n - \frac{f(x_n)}{f'(x_n)}}$
1	1	-1	2	1.5
2	1.5	0.25	3	1.41666666667
3	1.41666666667	0.00694444444	2.83333333333	1.41421568627
4	1.41421568627	0.00000600730	2.82843137255	1.41421356237
5	1.41421356237			

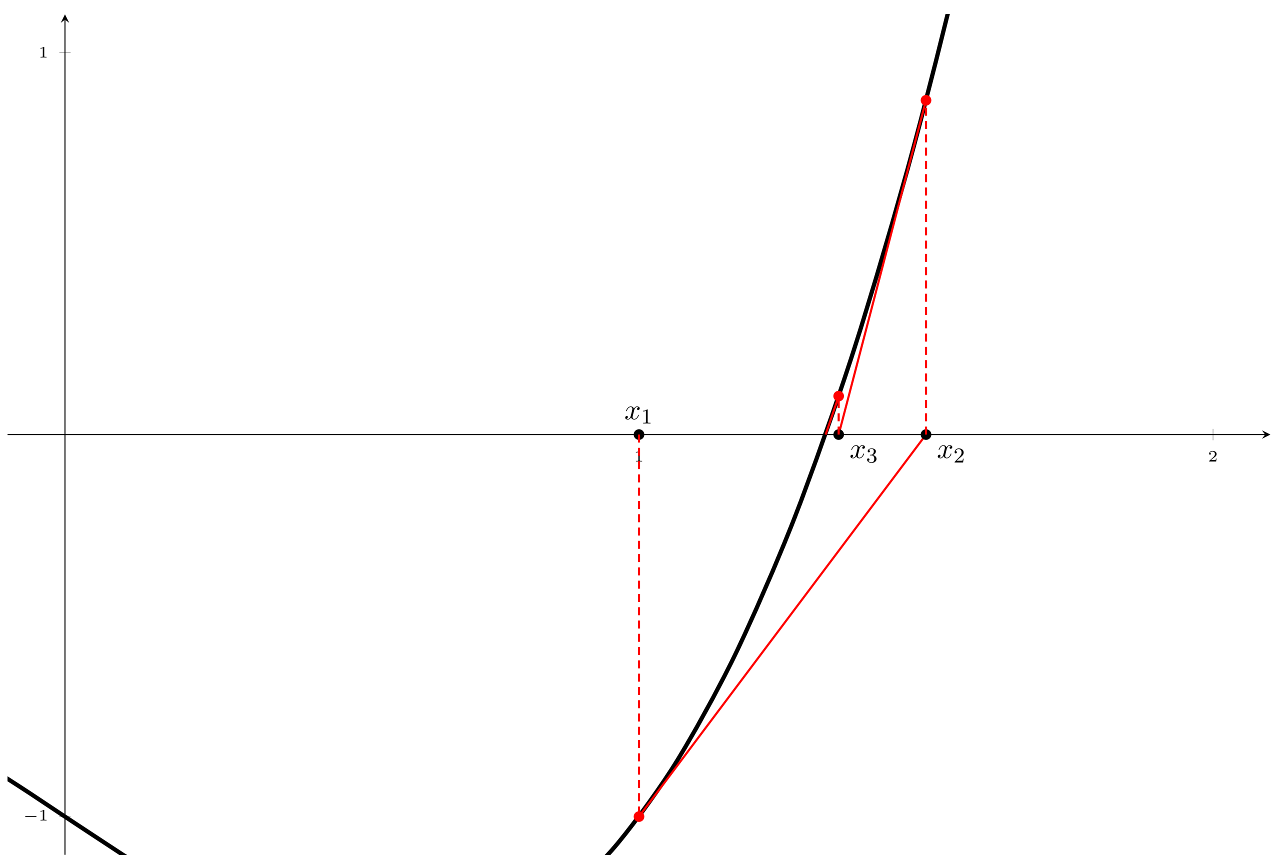


This process is unreasonably quick. In only 4 steps we accurately approximated $\sqrt{2}$ to 11 decimal places.

Example 4.8.2. Find the x-value where curve $y = x^3 - x$ crosses line $y = 1$.

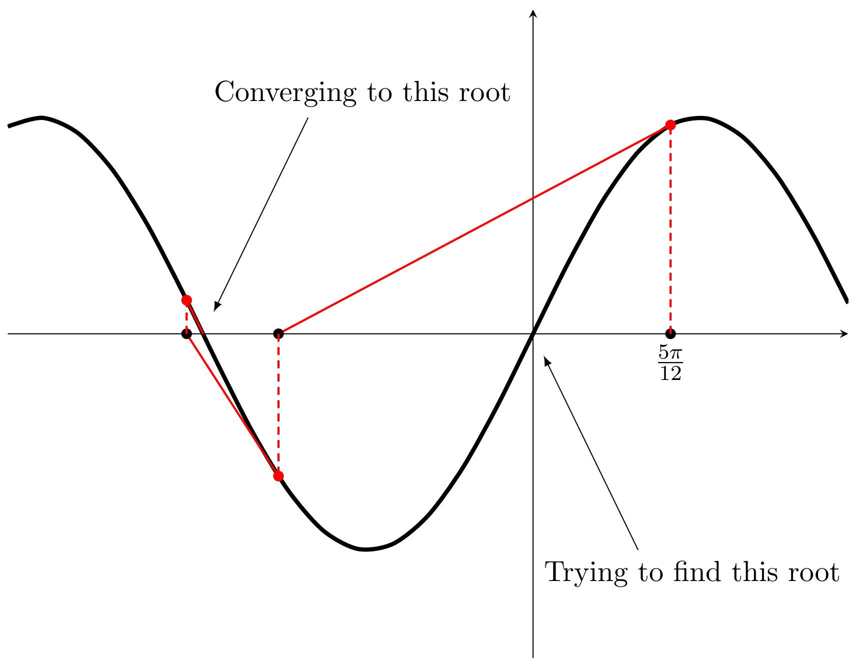
$x^3 - x = 1$ precisely when $x^3 - x - 1 = 0$. Set $f(x) = x^3 - x - 1$, and recall $f'(x) = 3x^2 - 1$. Since $f(1) = -1$ and $f(2) = 5$, so root occurs in interval $(1, 2)$. Guess $x_1 = 1$.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	2	1.5
2	1.5	0.875	5.75	1.34783
3	1.34783	0.10068	4.44991	1.32520
4	1.32520	0.00206	4.26847	1.32472
5	1.32472			



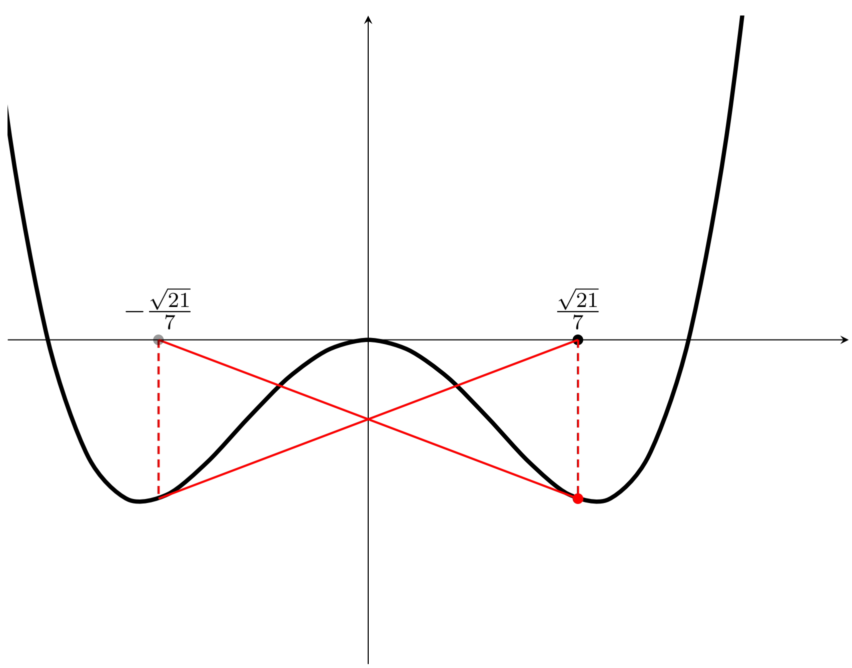
In fact, without rounding $x_5 = 1.324717957\ldots$, accurate to 8 decimal places.

Example 4.8.3. Draw the first few iterations of Newton's Method for $f(x) = \sin(x)$ with initial guess $x_1 = \frac{5\pi}{12}$. What do you notice?



It's important that our initial guess x_1 be as close as possible to the zero we want. If it is too far away, Newton's Method may converge to the wrong zero.

Example 4.8.4. Draw the first few iterations of Newton's Method for $f(x) = 4x^4 - 4x^2$ with initial guess $x_1 = \frac{\sqrt{21}}{7}$. What do you notice?

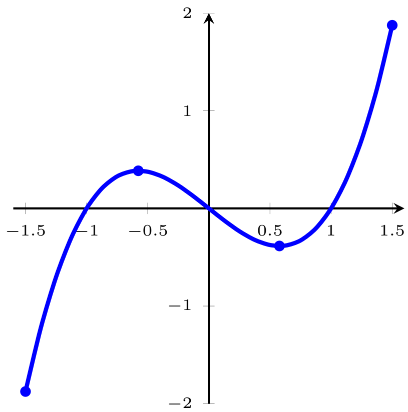


If we're very unfortunate, we may also end up in an infinite loop where the process never converges. Once again, we need to make sure that we pick x_1 very close to the zero we're seeking.

4.1 Minimum and Maximum Values

Definition. Let c be a number in the domain D of a function f. We say that $f(c)$ is an **absolute minimum** (or **global minimum**) value if, for all x in D, $f(c) \leq f(x)$. We say that $f(c)$ is an **absolute maximum** (or **global maximum**) value if, for all x in D, $f(c) \geq f(x)$. The minimum and maximum values of f are known as **extreme values** or just **extrema**.

Example 4.1.1. Consider the function $f(x) = x^3 - x$ where $-1.5 \leq x \leq 1.5$. Using a graph of the function, determine the global minimum and maximum values.



The absolute minimum occurs at $x =$ $\underline{\ -1.5\ }$ and the value is $\underline{\ -1.875\ }$.

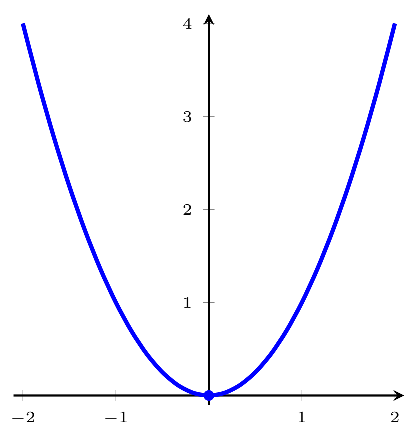
The absolute maximum occurs at $x =$ $\underline{\ 1.5\ }$ and the value is $\underline{\ 1.875\ }$.

In the previous example, the two points $\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{2}{3\sqrt{3}}\right)$ are not global extrema, but would be if we restricted our focus to just points around them.

Definition. Given f and c as in the previous definition, the number $f(c)$ is a **local minimum** (or **relative minimum**) value of f if $f(c) \leq f(x)$ for all x in an open interval containing c. The number $f(c)$ is a **local maximum** (or **relative maximum**) value of f if $f(c) \geq f(x)$ for all x in an open interval containing c. Local maxima/minima are sometimes referred to as **local extrema**.

Remark. Following along with your book's convention, local extrema will only occur on the *interior* of an interval, and never at the endpoints.

Example 4.1.2. What is the local minimum value of $f(x) = x^2$? What is the absolute minimum value of $f(x) = x^2$?

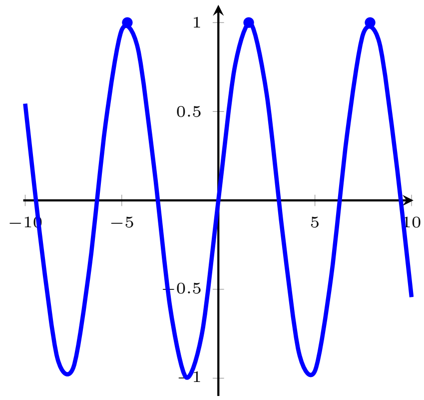


The local minimum occurs at $x =$ $\underline{\ 0\ }$ and the value is $\underline{\ 0\ }$.

The absolute minimum occurs at $x =$ $\underline{\ 0\ }$ and the value is $\underline{\ 0\ }$.

This last example indicates to us that local minima/maxima and absolute minima/maxima are not mutually exclusive. Indeed, they may be the same.

Example 4.1.3. What is the local maximum value of $g(x) = \sin x$? What is the absolute maximum value?



The local minimum occurs at $x = \underline{\quad 2n\pi + \frac{\pi}{2} \quad}$ and the value is $\underline{\quad 1 \quad}$.

The absolute minimum occurs at $x = \underline{\quad 2n\pi + \frac{\pi}{2} \quad}$ and the value is $\underline{\quad 1 \quad}$.

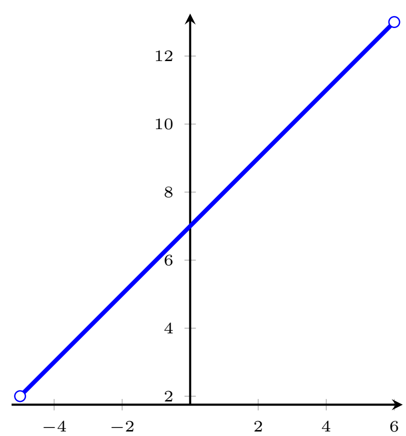
This last example indicates that the local and absolute extrema may be achieved infinitely many times.

As well, it may be that we have several different local maxima/minima (e.g., $y = x\cos x$), or none at all (e.g, $y = 5x + 7$). Similarly, absolute maxima/minima need not exist at all.

The following result tells us when absolute maxima/minima exist.

Theorem 4.1.4 (Extreme Value Theorem). *If f is continuous on a closed interval $[a, b]$, then there are numbers c and d in the interval $[a, b]$ so that f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$.*

Example 4.1.5. To see why the closed interval condition is required, consider the function $f(x) = x+7$ on the open interval $(-5, 6)$.

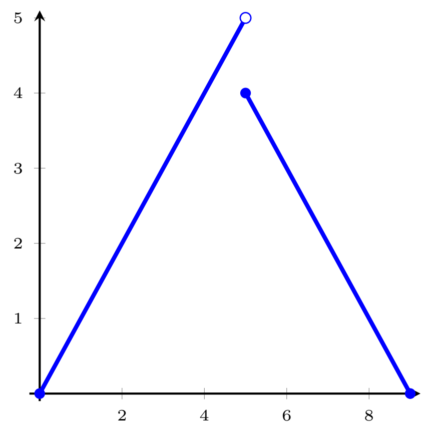


If there were a maximum value, it would be at 13 which occurs at $x = 6$, but 6 is not in the domain, so f does not attain an absolute maximum. By a similar argument, f does not attain an absolute minimum value.

Example 4.1.6. To see why the continuity is required, consider the following piecewise function with domain $[0, 9]$.

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 5 \\ 9 - x & \text{if } 5 \le x \le 9 \end{cases}$$

Certainly this function has an absolute minimum value of 0 (occurring at both $x = 0$ and $x = 9$), but it does not have an absolute maximum value.

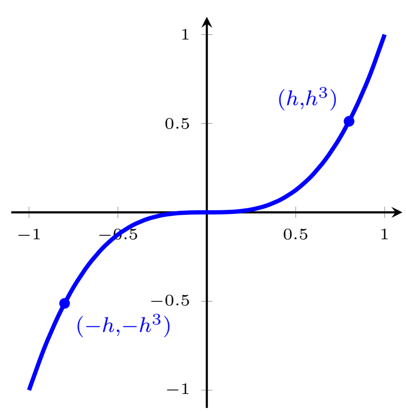


Now notice that local extrema occur at "peaks" and "valleys" on the graph of a function. The tangent lines appear horizontal at these points. Indeed, they are, and the following theorem tells us when local minima/maxima may exist.

Theorem 4.1.7. *If $f(c)$ is a local minimum or maximum and $f'(c)$ exists, then $f'(c) = 0$.*

Remark. It is important to note the logical implication of this theorem. Just because $f'(c) = 0$ or $f'(c)$ does not exist does not mean that $f(c)$ is an local minimum/maximum, it only give us *potential candidates* for local minima/maxima.

Example 4.1.8. Consider the function $f(x) = x^3$. Then $f'(x) = 3x^2 = 0$ precisely when $x = 0$. However, notice that 0 is not a local minimum or maximum because $(x - h)^3 < x^3 < (x + h)^3$ for all nonzero h, and in particular, $(-h)^3 < 0 < h^3$.



Although it may not correspond to a local minimum or maximum, we still give a name to x-values where the derivative is 0.

Definition. A **critical number** or **critical point** of a function f is the number c in the domain of f where either $f'(c) = 0$ or $f'(c)$ does not exist.

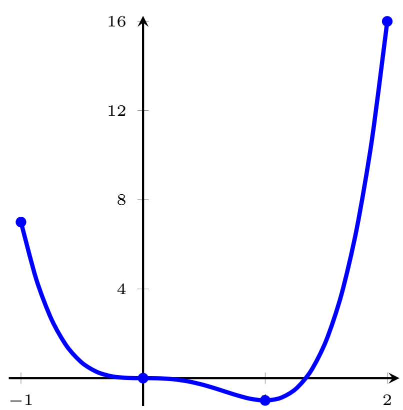
Procedure for finding extrema on a closed interval $[a, b]$:

1. Find the critical points of f in (a, b).
2. Evaluate f at each critical point in (a, b).
3. Evaluate f at each endpoint of the interval; i.e. find $f(a)$ and $f(b)$.
4. The smallest and largest values from the two previous steps are the absolute maximum and minimum, respectively.

Example 4.1.9. Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

Since $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$ we see that we have critical points at $x = 0$ and $x = 1$. We thus create the following table of values.

x	$f(x)$
-1	7
0	0
1	-1
2	16



The absolute minimum occurs at $x =$ __1__ and the value is __-1__.

The absolute minimum occurs at $x =$ __2__ and the value is __16__.

4.2 The Mean Value Theorem

Last time we talked about when local maxima and minima may exist, and we determined that critical points (in particular, zero derivatives) played a crucial roll. The following also gives us a criterion in which to check when critical points may exist.

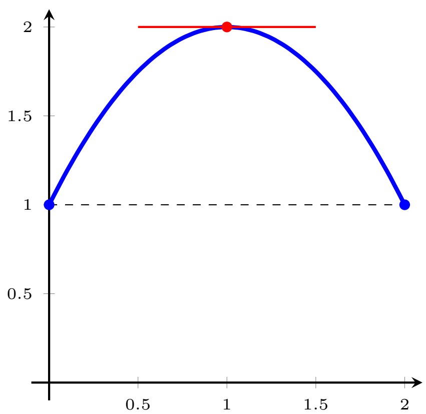
Theorem 4.2.1 (Rolle's Theorem)**.** *Suppose that f is*

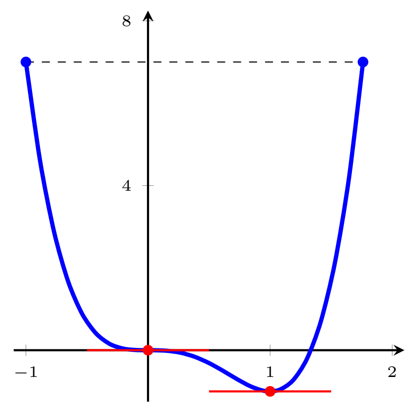
1. *continuous on the closed interval $[a, b]$,*
2. *differentiable on the open interval (a, b),*
3. *and satisfies $f(a) = f(b)$.*

Then there is a number c in (a, b) so that

$$f'(c) = 0.$$

Intuitively, this says that a function that starts and ends at the same value must have a zero derivative at some point in that interval.





Example 4.2.2. Prove that $x^5 + x^3 + x + 1$ has only one real root.

Let $f(x) = x^5 + x^3 + x + 1$. $f(-1) < 0$ and $f(0) > 0$ so by IVT there is an x-value, c, in $(-1, 0)$ where $f(c) = 0$.
Suppose now it has 2 (or more) real roots, a and b. Then $f(a) = f(b) = 0$. Since $f(x)$ is a polynomial function, Rolle's theorem applies, hence there must be some x-value, c, in (a, b) where $f'(c) = 0$. However,

$$f'(x) = 5x^4 + 3x^2 + 1$$

is positive for every x-value, which is absurd. We conclude that $f(x)$ cannot cross the x-axis more than once, and therefore the polynomial $x^5 + x^3 + x + 1$ has exactly one real root.

Example 4.2.3. A person throws a ball vertically into the air. 4 seconds later, that same person catches it. Why must there be some time at which the ball was at rest in these 4 seconds?

The ball moves continuously, and as we've seen before, the position function $s(t)$ is differentiable. By Rolle's Theorem, there is some point in time t in $(0, 4)$ for which $v(t) = s'(t) = 0$, hence there is an instant in which the ball is at rest.

This theorem can be generalized by imagining what would happen if we took the graph of our function f on $[a, b]$ and rotated it. The horizontal tangent line would rotate as well and would still be parallel to the secant line between $(a, f(a))$ and $(b, f(b))$. The following theorem states this quite concretely.

Theorem 4.2.4 (Mean Value Theorem)**.** *Let f be a function that satisfies the following hypotheses:*

1. *f is continuous on $[a, b]$.*
2. *f is differentiable on (a, b).*

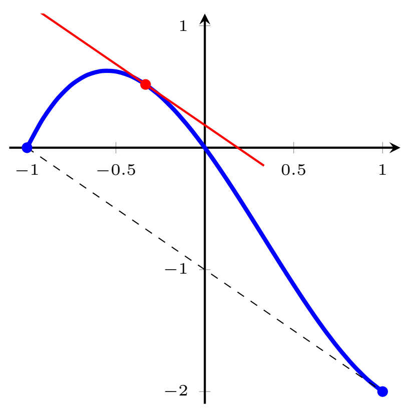
Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The number on the right-hand side of the equals sign is the slope of the line between $(a, f(a))$ and $(b, f(b))$. This slope is sometimes **average slope** or **mean slope**, which is where the theorem gets its name.

Remark. In the case where $f(a) = f(b)$, the Mean Value Theorem is exactly Rolle's Theorem

Example 4.2.5. Consider the function $f(x) = x^3 - x^2 - 2x$ on $[-1, 1]$. Find the value(s) of c that satisfies the conclusion of the Mean Value Theorem (MVT).



We first note that since $f(x)$ is a polynomial, it is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, so we can apply the Mean Value Theorem. Now, we have

$$f'(x) = 3x^2 - 2x - 2,$$

and

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2 - 0}{2} = -1.$$

By the Mean Value Theorem there exists some number c in $(-1, 1)$ so that

$$\begin{aligned} 3c^2 - 2c - 2 &= -1 \\ 3c^2 - 2c - 1 &= 0 \\ (3c + 1)(c - 1) &= 0, \end{aligned}$$

so $c = -\frac{1}{3}$. ($c \neq 1$ since the Mean Value Theorem only gives us c-values in $(-1, 1)$.)

Example 4.2.6. Does there exist a continuous function f such that $f(1) = 6$, $f(7) = 9$, and $f'(x) \geq 2$ for all x in $(1, 7)$?

No. Since f satisfies the conditions of the mean value theorem, there must exist c in the interval $(1, 7)$ such that

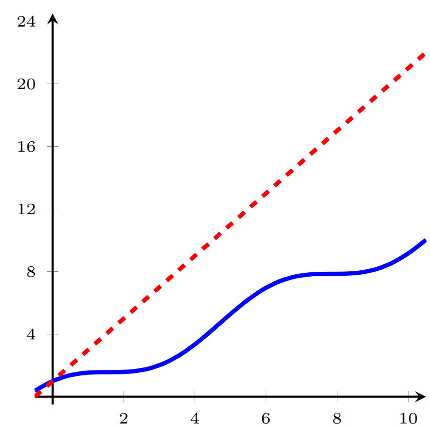
$$f'(c) = \frac{f(7) - f(1)}{7 - 1} = \frac{9 - 6}{6} = \frac{3}{6} = \frac{1}{2}.$$

As such, it is impossible for $f'(x) \geq 2$ for all c in $(1, 7)$.

Example 4.2.7. Suppose f is some differentiable function with $f(0) = 1$ and $f'(x) \leq 2$ for all x values. How large can $f(10)$ possibly be?

Since f is differentiable, then by the Mean Value Theorem, there must be some c in $(0, 10)$ for which

$$f'(c) = \frac{f(10) - f(0)}{10 - 0}$$
$$\implies f(10) = f(0) + 10f'(c) \leq 1 + 10(2) = 21$$



Proposition 4.2.8. *If $f'(x) = 0$ for all x-values, then $f(x) = const.$*

Proof. If the function were not constant, there would be two x-values, a and b, where $f(a) \neq f(b)$, and by the Mean Value Theorem, there would be some c-value in (a, b) where $f'(c) \neq 0$, which is absurd. □

Example 4.2.9. Determine the value of $\tan^{-1}(x) + \cot^{-1}(x)$. Let $f(x) = \tan^{-1}(x) + \cot^{-1}(x)$. This function is differentiable and

$$f'(x) = \frac{1}{1 + x^2} + \frac{-1}{1 + x^2} = 0.$$

By the above Proposition, $f(x)$ must be a constant. We determine this constant by checking the value when $x = 1$ and get

$$f(x) = f(1) = \tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

4.3 How Derivatives Affect the Shape of a Graph

4.3.1 What Does f' Say About f?

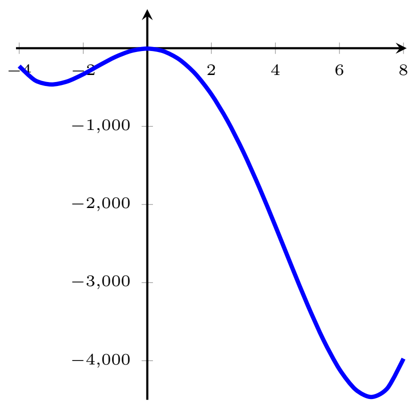
Notice that, for a given graph of a function, the slope of the tangent line is positive when the function values are increasing, and the slope of the tangent line is negative when the function values are decreasing. Derivatives allow us to figure out where these intervals of increase/decrease occur.

Proposition 4.3.1 (Increasing/Decreasing Test)**.**

a. *If $f'(x) > 0$ on an interval (a, b), then f is increasing on the interval (a, b).*

b. *If $f'(x) < 0$ on an interval (a, b), then f is decreasing on the interval (a, b).*

Example 4.3.2. Find the intervals of increase and decrease for the function $f(x) = 3x^4 - 16x^3 - 126x^2 - 5$.



First, we take the derivative.

$$f'(x) = 12x^3 - 48x^2 - 252x = 12x(x + 3)(x - 7)$$

By the Increasing/Decreasing test, it suffices to find intervals for which $f'(x) < 0$ and $f'(x) > 0$. This means that the intervals to test occur between critical points. Since $f'(x) = 0$ for $x = -3, 0, 7$, we consider the four intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 7)$, and $(7, \infty)$.

By the Intermediate Value Theorem, it suffices to check a single value in each interval to determine the derivative's sign on each interval. We'll check -4, -1, 1, and 8.

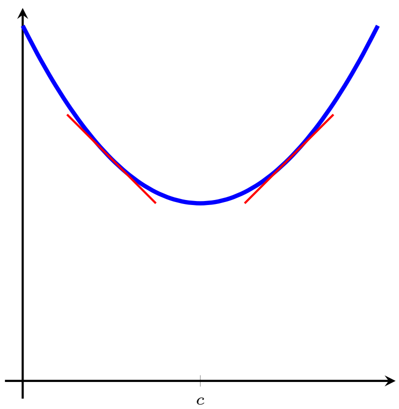
$$f'(-4) = -528$$
$$f'(-1) = 192$$
$$f'(1) = -288$$
$$f'(8) = 1056$$

Interval	$f'(x)$	f
$(-\infty, -3)$	$-$	decreasing
$(-3, 0)$	$+$	increasing
$(0, 7)$	$-$	decreasing
$(7, \infty)$	$+$	increasing

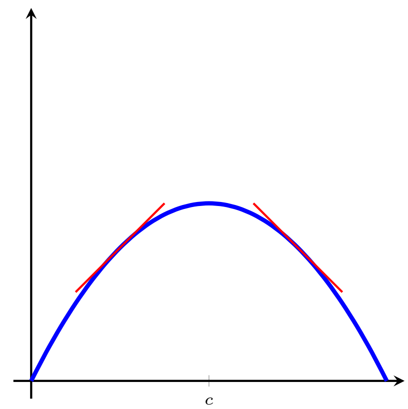
What we notice from this last example is that our local maxima/minima occur between sign changes for our derivative. This leads us to the following test for local extrema.

Theorem 4.3.3 (First Derivative Test)**.** *Suppose that f is continuous and that c is a critical point of f.*

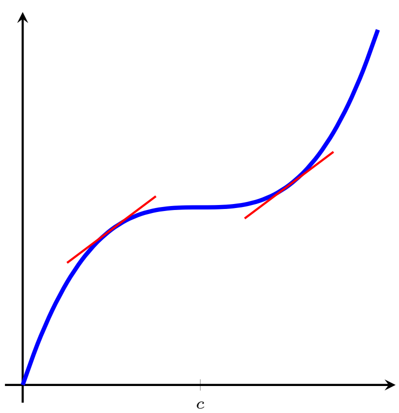
- ***a.*** *If f' changes from positive to negative at c, then f has a local maximum at c.*
- ***b.*** *If f' changes from negative to a positive at c, then f has a local minimum at c.*
- ***c.*** *If f' does not change sign at c, then f has no local minimum or maximum at c.*



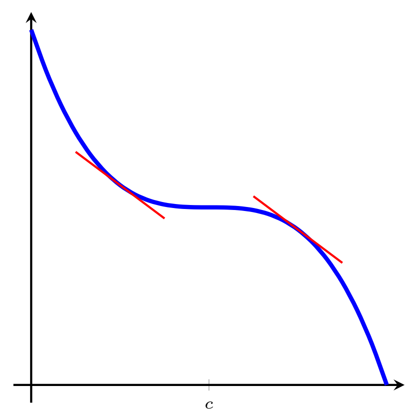
Local minimum



Local maximum



No local maximum/minimum



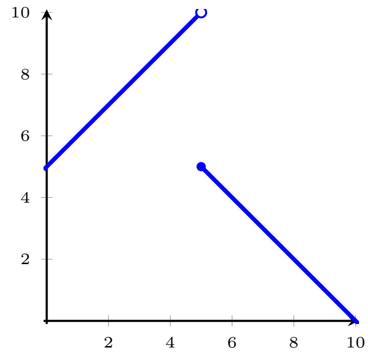
No local maximum/minimum

Example 4.3.4. Use the First Derivative Test to find minimum and maximum values of the function f in Example 4.3.2.

From the chart, we see that $f'(x)$ changes from negative to positive at -3, so -3 is a local minimum with value $f(-3) = -464$. Also, $f'(x)$ changes from positive to negative at 0, so 0 is a local maximum with value $f(0) = -5$. Finally, $f'(x)$ changes from negative to positive at 7, so 7 is a local minimum with value $f(7) = -4464$.

Example 4.3.5. Continuity is important for the First Derivative Test. To see why, consider the function

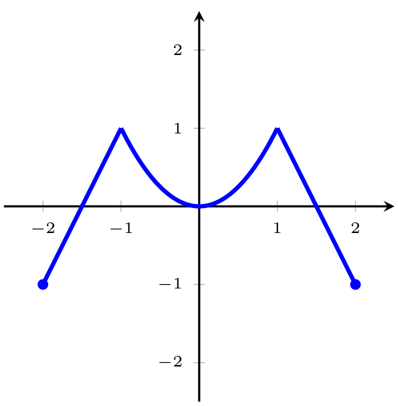
$$f(x) = \begin{cases} x+5 & \text{when } x < 5, \\ 10-x & \text{when } x \geq 5. \end{cases}$$



There is a critical point at $x = 5$ and the derivative changes from positive to negative at $x = 5$, but this is not a local maximum.

Example 4.3.6. Find the local and global extrema of the function

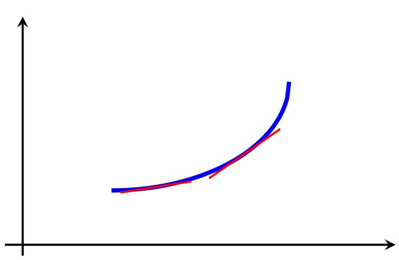
$$f(x) = \begin{cases} 3+2x & \text{when } -2 \leq x \leq -1, \\ x^2 & \text{when } -1 < x < 1, \\ 3-2x & \text{when } 1 \leq x \leq 2. \end{cases}$$

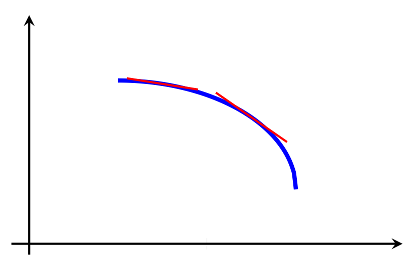


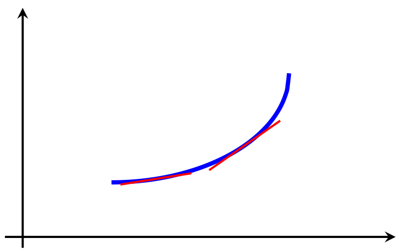
Endpoints at $x = -2$, $x = 2$.
Critical points at $x = -1$, $x = 0$, $x = 1$.
Local max at $x = -1$, $x = 1$ by First Derivative Test.
Local min at $x = 0$ by First Derivative Test.
Global max at $x = -1$, $x = 1$.
Global min at $x = -2$, $x = 2$.

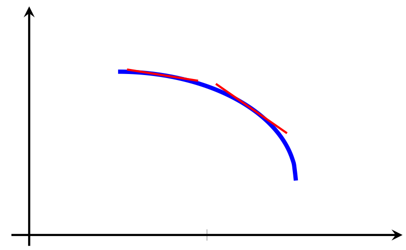
4.3.2 What Does f'' Say About f'?

In the two graphs below, both functions have a positive first derivative, and yet there's something fundamentally different about them.









Definition. If the graph of f lies above all of its tangents on an interval, I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Since curves can have multiple intervals on which they are concave upward or concave downward, we give a special name to these points where the concavity changes.

Definition. A point on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward (or *vice-versa*) at that point.

Using the second derivative, we get the following test for concavity.

Proposition 4.3.7 (Concavity Test)**.**

a. *If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.*

b. *If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.*

Example 4.3.8. Find the intervals of concavity for function f in Example 4.3.2. First, we take the second derivative.

$$f''(x) = 36x^2 - 96x - 252$$

By the Concavity test, it suffices to find intervals for which $f''(x) < 0$ and $f''(x) > 0$. This means that the intervals to test occurs between x-values for which $f''(x) = 0$. The Quadratic Formula gives us that $f''(x)$ has roots at $x = \frac{4}{3} - \frac{\sqrt{79}}{3} \approx -1.629$ and at $x = \frac{4}{3} + \frac{\sqrt{79}}{3} \approx 4.296$. Thus the intervals to consider are $(-\infty, -1.629)$, $(-1.629, 4.296)$, and $(4.296, \infty)$.

By the Intermediate Value Theorem, it suffices to check a single value in each interval to determine the second derivative's sign on each interval. We'll check -2, 0, and 5.

$$f''(-2) = 84$$
$$f''(0) = -252$$
$$f''(5) = 168$$

Interval	$f'(x)$	f
$\left(-\infty, \frac{4}{3} - \frac{\sqrt{79}}{3}\right)$	$+$	concave up
$\left(\frac{4}{3} - \frac{\sqrt{79}}{3}, \frac{4}{3} + \frac{\sqrt{79}}{3}\right)$	$-$	concave down
$\left(\frac{4}{3} + \frac{\sqrt{79}}{3}, \infty\right)$	$+$	concave up

Remark. By the Intermediate Value Theorem, we can deduce that inflection points must occur where $f''(x) = 0$. Like local minima/maxima, this condition alone is not enough to test whether points are inflection points (consider $f(x) = x^4$ and $c = 0$).

Going back to maxima/minima, the second derivative and concavity give us the following useful test.

Theorem 4.3.9 (Second Derivative Test)**.** *Suppose f'' is continuous near c.*

a. *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*

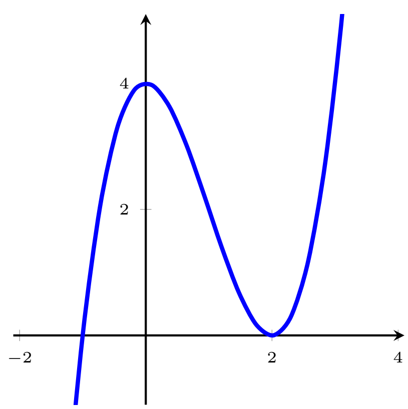
b. *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Remark. The second derivative test is inconclusive when $f''(c) = 0$ or $f''(c)$ does not exist. There may be a local maximum, local minimum, or neither at c. In these instances, we may have to resort to using the First Derivative Test

Example 4.3.10. Use the second derivative test to find relative extrema for the function f in Example 4.3.2.

We know that $f'(c) = 0$ for $c = -3, 0, 7$. Since $f''(-3) = 360 > 0$, we have that f has a local minimum at 3 by the Second Derivative test. Similarly, $f''(0) = -252 < 0$, so we have that f has a local maximum at 0. Lastly, $f''(7) = 840 > 0$, so we have that f has a local minimum at 7. This agrees with the results from the First Derivative Test as in Example 4.3.4.

Example 4.3.11. Use the second derivative test to find relative extrema for the function g given by $g(x) = x^3 - 3x^2 + 4$.



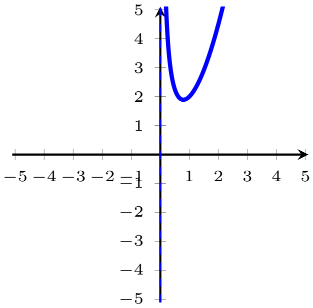
Relative extrema occur at critical points, and since g is differentiable, this happens when

$$0 = g'(x) = 3x^2 - 6x = 0$$
$$\Rightarrow x = 0, 2.$$

The second derivative is $g''(x) = 6x - 6$. Since $g''(0) < 0$, there is a local maximum at $x = 0$, and since $g''(2) > 0$, there is a local minimum at $x = 2$.

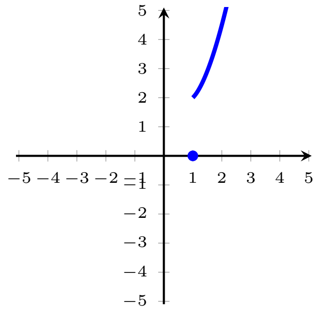
Example 4.3.12. Find the locations and values of all local and global extrema for the function $A(r) = r^2 + \dfrac{1}{r}$ on the given domain:

a. $(0, \infty)$



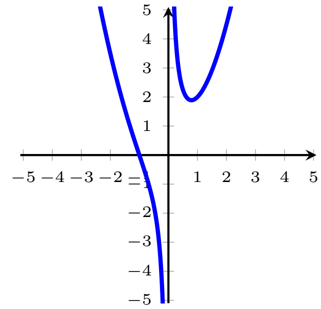
Local max: None
Local min: $(x, y) \approx (0.79, 1.89)$
Global max: None
Global min: $(x, y) \approx (0.79, 1.89)$

b. $[1, \infty)$



Local max: None
Local min: None
Global max: None
Global min: $(x, y) = (1, 2)$

c. $[-1, 0) \cup (0, 1]$



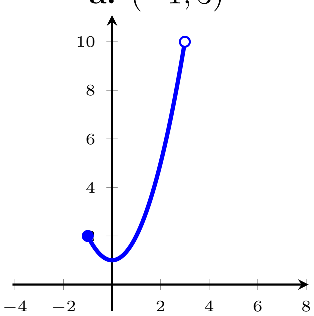
Local max: None
Local min: $(x, y) \approx (0.79, 0.89)$
Global max: None
Global min: None

Example 4.3.13. Find the locations and values of all local and global extrema for the function

$$g(x) = \begin{cases} x^2 + 1 & \text{when } x < 3 \\ (x-4)^2 + 2 & \text{when } x \geq 3 \end{cases}$$

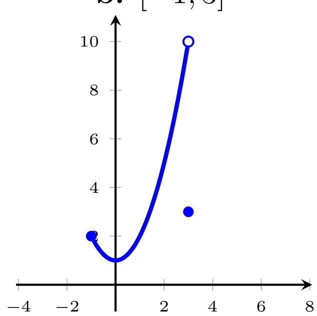
on the given domain:

a. $(-1, 3)$



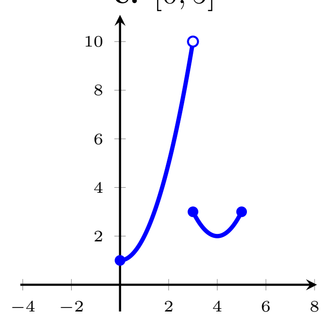
Local max: None
Local min: $(x, y) = (0, 1)$
Global max: None
Global min: $(x, y) = (0, 1)$

b. $[-1, 3]$



Local max: None
Local min: $(x, y) = (0, 1)$
Global max: None
Global min: $(x, y) = (0, 1)$

c. $[0, 5]$



Local max: None
Local min: $(x, y) = (4, 2)$
Global max: None
Global min: $(x, y) = (0, 1)$

4.5 Summary of Curve Sketching

We should collect the following information to sketch a curve. We note that the list below is intended to be a checklist and may not be relevant for every curve.

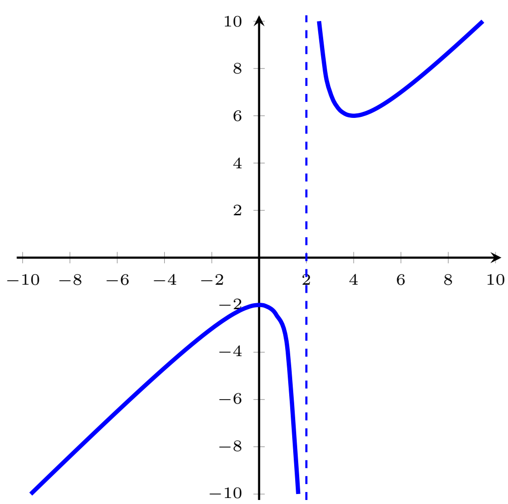
- **a.** Domain
- **b.** Intercepts
- **c.** Symmetry (even/odd)
- **d.** Asymptotes
- **e.** Intervals of Increase/Decrease
- **f.** Local minima/maxima and values
- **g.** Concavity and inflection points

Example 4.5.1. Sketch a graph of $y = f(x)$ where $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

- **a.** Domain: $(-\infty, 2) \cup (2, \infty)$.
- **b.** x-intercepts: None
  y-intercept: $(0, -2)$
- **c.** Horizontal asymptotes: None
  Vertical asymptotes: $x = 2$
- **d.** Symmetry: None (f is neither even nor odd)
- **e.** $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$. Critical points at $x = 0, 4$
- **f.** Local max: $(0, -2)$ (First Derivative Test)
  Local min: $(4, 6)$ (First Derivative Test)
- **g.** $f''(x) = \dfrac{8}{(x-2)^3}$. Inflection points: none.

Interval	$f'(x)$	f
$(-\infty, 0)$	$+$	Increasing
$(0, 2)$	$-$	Decreasing
$(2, 4)$	$-$	Decreasing
$(4, \infty)$	$+$	Increasing

Interval	$f''(x)$	f
$(-\infty, 2)$	$-$	Concave Down
$(2, \infty)$	$+$	Concave Up



Example 4.5.2. Sketch a graph of $y = \dfrac{e^x}{x^2}$.

a. Domain: $(-\infty, 0) \cup (0, \infty)$.

b. x-intercepts: None
y-intercept: None

c. Horizontal asymptotes: $y = 0$ (as $x \to -\infty$ only)
Vertical asymptotes: $x = 0$

d. Symmetry: None (f is neither even nor odd)

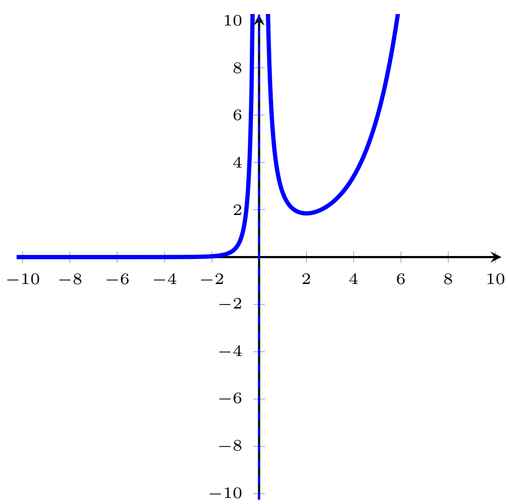
e. $f'(x) = \dfrac{e^x(x-2)}{x^3}$. Critical point at $x = 2$

f. Local min: $\left(2, \frac{e^2}{4}\right) \approx (2, 1.85)$ (First Derivative Test)

Interval	$f'(x)$	f
$(-\infty, 0)$	$+$	Increasing
$(0, 2)$	$-$	Decreasing
$(2, \infty)$	$+$	Increasing

g. $f''(x) = \dfrac{e^x(x^2 - 4x + 6)}{x^4}$. Inflection points: none.

Interval	$f''(x)$	f
$(-\infty, 0)$	$+$	Concave Up
$(0, \infty)$	$+$	Concave Up



Example 4.5.3. Sketch a graph of $y = 8x^{1/3} - 2x$.

a. Domain: $(-\infty, \infty)$.

b. x-intercepts: $(-8, 0)$, $(0, 0)$, $(8, 0)$
y-intercept: $(0, 0)$

c. Horizontal asymptotes: None
Vertical asymptotes: None

d. Symmetry: f is odd

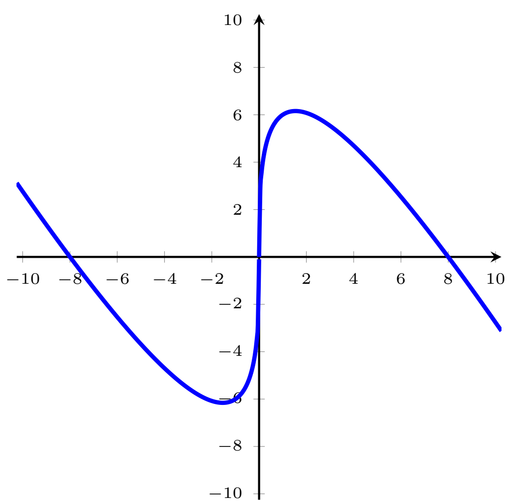
e. $f'(x) = \dfrac{8}{3x^{2/3}} - 2$. Critical points at $x = 0, \pm\dfrac{8}{3\sqrt{3}}$

Interval	$f'(x)$	f
$(-\infty, -\frac{8}{2\sqrt{3}})$	$-$	Decreasing
$(-\frac{8}{2\sqrt{3}}, 0)$	$+$	Increasing
$(0, \frac{8}{2\sqrt{3}})$	$+$	Increasing
$(\frac{8}{2\sqrt{3}}, \infty)$	$-$	Decreasing

f. Local min: $\left(-\dfrac{8}{2\sqrt{3}}, -\dfrac{32}{3\sqrt{3}}\right) \approx (-6.16, -1.54)$ (First Derivative Test)
Local min: $\left(\dfrac{8}{2\sqrt{3}}, \dfrac{32}{3\sqrt{3}}\right) \approx (6.16, 1.54)$ (First Derivative Test)

Interval	$f''(x)$	f
$(-\infty, 0)$	$+$	Concave Up
$(0, \infty)$	$-$	Concave Down

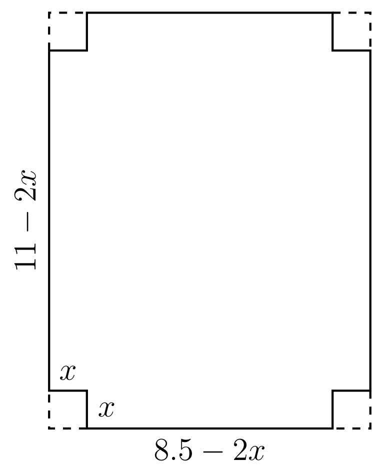
g. $f''(x) = \dfrac{-16}{9x^{5/3}}$. Inflection points: $(0, 0)$



4.7 Optimization Problems

Another important application of derivatives is in optimization problems. When doing these types of problems, it's you should keep in mind the pieces of information that you'll need - clearly defined variables, a function to be optimized, and any constraints.

Example 4.7.1. A lidless box is formed from an $8.5'' \times 11''$ sheet of paper by cutting a square from each corner and folding the sides up. What dimensions of the corner squares will maximize the volume of the box?



Constraints: $0\,\text{in} \leq x \leq 4.25\,\text{in}$

By cutting the sheet of paper as described, the box will have length $11 - 2x$ in, width $8.5 - 2x$ in, and height x in. As such, the volume of this box is given by

$$V(x) = (11 - 2x)(8.5 - 2x)x = 4x^3 - 39x^2 + 93.5x$$

By the Extreme Value Theorem, this function does obtain a maximum and a minimum on the x-interval $[0, 4.25]$. Intuitively, we know that the endpoints both occur to global minima (because the volume would be 0), and so the maximum volume must occur at a local maximum on the interior of this interval.

$$V'(x) = 12x^2 - 78x + 93.5 = 0 \quad \Rightarrow x \approx 1.59\,\text{in}, 4.91\,\text{in}.$$

Since the latter answer is outside of the allowable x-values, the only critical point occurs when $x \approx 1.59$, and a quick check with the First Derivative Test indicates that this is indeed a local maximum.

Thus, when the corner square is approximately $1.59\,\text{in} \times 1.59\,\text{in}$, the volume of the box is maximized, and the volume is approximately $66.15\,\text{in}^3$.

Example 4.7.2. *Punkin chunkin* is a timeless event where people fire small pumpkins out of specially-designed cannons and compete to see whose pumpkin flies the farthest. The average mass of a punkin chunkin pumpkin is 4.5 kg and a standard cannon can fire a pumpkin with initial velocity 150 m/s. Ignoring the air resistance, the horizontal range R of a mass with initial velocity v fired at an angle θ above the ground is given by

$$R = \frac{v^2}{g}\sin(2\theta)$$

where $g = 9.8\,\text{m/s}$ is the acceleration due to gravity. Find the maximum range of a punkin chunkin pumpkin.

Constraints: $0 \le \theta \le \frac{\pi}{2}$

With our constants, we have

$$R(\theta) = \frac{150^2}{9.8}\sin(2\theta)$$

and by the Extreme Value Theorem, R must attain both a minimum and a maximum on this θ-interval $[0, \frac{\pi}{2}]$. Intuitively we know that at the extremes of this domain, the range is 0, so we need only find a local maximum on the interior.

$$R'(\theta) = \frac{2\cdot 150^2}{9.8}\cos(2\theta) = 0 \qquad \Rightarrow \qquad \theta = \frac{\pi}{4}$$

A quick check of the First Derivative Test suggests that a local maximum occurs at $\theta = \frac{\pi}{4}$, and hence this must be a global maximum.

Therefore, the maximum range of the pumpkin is $R(\frac{\pi}{4}) = \frac{150^2}{9.8}\sin(\frac{\pi}{2}) = 2295.92\,\text{m}$.

Example 4.7.3. Find two positive numbers whose product is 169 and whose sum is a minimum.

First we'll call these numbers x and y. We're looking to find the minimum of $x + y$, subject to the constraints that $xy = 169$ and x, y in $(0, \infty)$. Rearranging our constraint, we see that $y = \frac{169}{x}$, so we can rewrite our sum as the function

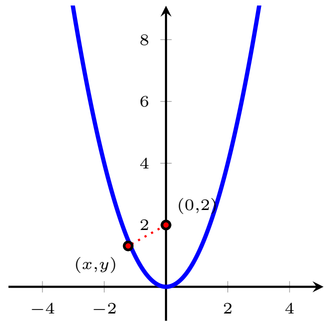
$$f(x) = x + y = x + \frac{169}{x}.$$

To find the minimum on the interval $(0, \infty)$, we take the derivative and set it equal to 0.

$$\begin{aligned} f'(x) = 1 - \frac{169}{x^2} &= 0 \\ 1 &= \frac{169}{x^2} \\ x^2 &= 169 \\ \Rightarrow \quad x &= 13. \end{aligned}$$

Plugging this back into our constraint equation, $y = \frac{169}{x} = \frac{169}{13} = 13$. So, the pair of numbers whose product is 169 and whose sum is a minimum is $x = 13$, $y = 13$.

Example 4.7.4. Find the point(s) on the graph $y = x^2$ that are closest to the point $(0, 2)$.



The distance between a point $P = (0, 2)$ and a point $Q = (x, y)$ is given by

$$d(P, Q) = \sqrt{x^2 + (y - 2)^2}$$

and this is the function we will ultimately try to minimize, subject to the constraint that Q must lie on the curve $y = x^2$. Setting $y = x^2$,

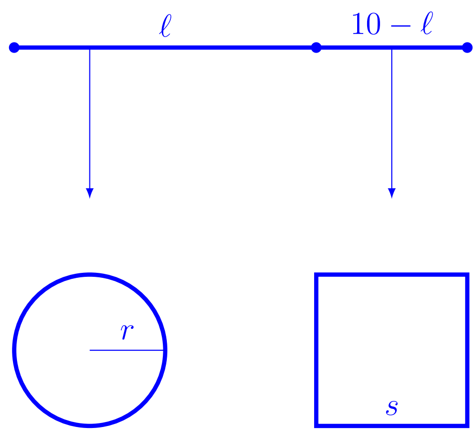
$$d(P, Q) = \sqrt{x^2 + (y - 2)^2} = \sqrt{x^2 + (x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}$$

which has domain $(-\infty, \infty)$. Taking the derivative with respect to x, we have

$$d'(P, Q) = \frac{4x^3 - 3x}{2\sqrt{x^4 - 3x^2 + 4}}$$

which has critical points at $x = 0$, $x = \pm\frac{1}{\sqrt{3}}$. Using the First Derivative Test, we see that $x = \pm\frac{1}{\sqrt{3}}$ are both local minima and $x = 0$ is a local maximum. Since d' is negative for $x < -\frac{1}{\sqrt{3}}$ and positive for $x > \frac{1}{\sqrt{3}}$, at least one of $x = \pm\frac{1}{\sqrt{3}}$ must correspond to global minimum. In fact, by checking the function values, both are global minima, and thus, the two points closest to $(0, 2)$ on the curve $y = x^2$ are $\left(-\frac{1}{\sqrt{3}}, \frac{1}{3}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{3}\right)$.

Example 4.7.5. A piece of wire 10 feet in length is cut into two pieces; one piece is bend into a square, the other rolled into a circle. Where, if anywhere, should the wire be cut to maximize the total area enclosed by the 2 objects?



Constraints:

- $0 \leq \ell \leq 10$
- $\ell = 2\pi r$
- $10 - \ell = 4s$

We're trying to maximize the total area enclosed, which is given by

$$A = \pi r^2 + s^2 = \pi \left(\frac{\ell}{2\pi}\right)^2 + \left(\frac{10 - \ell}{4}\right)^2$$

By the Extreme Value Theorem, there must exist a maximum, either at the endpoints, or at the critical points. To find the critical points, we differentiate

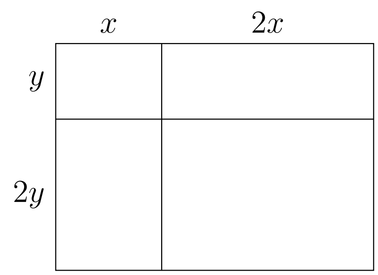
$$A'(\ell) = 0 \quad \Rightarrow \quad \ell = \frac{10\pi}{4 + \pi}.$$

Checking the area values, we have

ℓ	$A(\ell)$
$0\,\text{ft}$	$\frac{25}{4} = 6.25\,\text{ft}^2$
$\frac{10\pi}{4+\pi}\,\text{ft}$	$\frac{25}{4+\pi} \approx 3.50062\,\text{ft}^2$
$10\,\text{ft}$	$\frac{25}{\pi} = 7.95775\,\text{ft}^2$

So the maximum area occurs when we use the entire length of wire to construct the circle.

Example 4.7.6. A farmer has 360 feet of fencing with which to build the pen shown below. What is the maximum area that the farmer can enclose?



We're trying to optimize the area $A = (3x)(3y)$, subject to the constraints that $3(3x) + 3(3y) = 360$ and x, y in $[0, 40]$. Rearranging our constraint, we see that $y = 40 - x$, so we can rewrite our area as the function

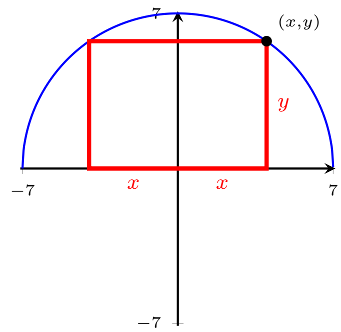
$$A(x) = (3x)(3y) = (3x)(120 - 3x) = 360x - 9x^2.$$

To find the maximum on the interval $[0, 40]$, we start by taking the derivative and setting it equal to 0.

$$\begin{aligned} A'(x) = 360 - 18x &= 0 \\ 360 &= 18x \\ \Rightarrow \quad x &= 20. \end{aligned}$$

By the Extreme Value Theorem, the maximum enclosed area must occur when $x = 0$, $x = 20$, or $x = 40$. Indeed, by comparing these values, we have that the maximum occurs at $x = 20$, and the maximum area enclosed is $3600\,\text{ft}^2$.

Example 4.7.7. A rectangle is bounded between the x-axis and the semicircle $y = \sqrt{49 - x^2}$. What length and width should the rectangle have so that its area is maximum?



Constraints:

- $0 \leq x \leq 7$
- $0 \leq y \leq 7$
- $y = \sqrt{49 - x^2}$

We're trying to maximize the area $A = (2x)(y)$ subject to the constraints above. We can rewrite our area function as

$$A(x) = (2x)(y) = 2x\sqrt{49 - x^2}.$$

Taking the derivative and setting it equal to 0, we see that

$$\begin{aligned} A'(x) = 2\sqrt{49 - x^2} + \frac{-2x^2}{\sqrt{49 - x^2}} = 02\sqrt{49 - x^2} &= \frac{2x^2}{\sqrt{49 - x^2}} \\ 2(49 - x^2) &= 2x^2 \\ 49 - x^2 &= x^2 \\ 2x^2 &= 49 \\ \Rightarrow \quad x &= \frac{7\sqrt{2}}{2} \approx 4.9497. \end{aligned}$$

This means that the maximum area occurs when $x = 0$, $x = \frac{7\sqrt{2}}{2}$, or $x = 7$. By plugging each of these three values into $A(x)$, we see that it occurs when $x = \frac{7\sqrt{2}}{2}$. So the dimensions of the rectangle are $7\sqrt{2}$ by $\frac{7\sqrt{2}}{2}$, or approximately 9.8995 by 4.9497.

Example 4.7.8. You have been asked to design a 1-liter ($1000\,\text{cm}^3$) cylindrical can. What dimensions of the can use the least material? Recall that the volume of a cylinder with radius r and height h has volume $V = \pi r^2 h$ and surface area $A = 2\pi rh + 2\pi r^2$.

Constraints:

- $\pi r^2 h = 1000$
- $r > 0$
- $h > 0$

We're trying to minimize the surface area equation, which can be rewritten as

$$A = 2\pi rh + 2\pi r^2 = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2.$$

Since

$$A'(r) = -\frac{2000}{r^2} + 4\pi r$$

and since $r > 0$, we have a single critical point when $r = \sqrt[3]{\frac{500}{\pi}}$ cm. By the first derivative test, we see that r corresponds to both a local and global minimum, hence the minimum surface area is

$$A\left(\sqrt[3]{\frac{500}{\pi}}\right) = 300\sqrt[3]{2\pi}\,\text{cm}^2 \approx 553.581\,\text{cm}^2.$$

4.9 Antiderivatives

In a totally-not-contrived scenario, suppose your friend throws a ball straight into the air, tells you the function of the ball's velocity $v(t)$, and asks you about the position of the ball at any given time. Since we know that velocity is the first derivative of the position function, the question boils down to finding some function $p(t)$ so that $v(t) = p'(t)$.

Definition. A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 4.9.1. Find an antiderivative F for the function $f(x) = x^3$.

Just keeping in mind the power rule, we can see that $F(x) = \frac{1}{4}x^4$ is an antiderivative for f.

Of course, because the derivative of a constant is 0, $F(x) = \frac{1}{4}x^4 + 100$ is also an antiderivative for f. This tells us that antiderivatives are not unique and can differ by a constant. In the totally-not-contrived scenario, the interpretation is this: you can tell what the position of the ball is exactly if you know what the initial height of your friend's throw was, and that could change depending on whether s/he was standing on top of a building, at sea level, or at the bottom of the Bingham Canyon Mine.

Proposition 4.9.2. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C,$$

where C is an arbitrary constant.

Example 4.9.3. Find the most general antiderivative of each of the following functions.

a. $f(x) = x^n$, $n \neq -1$

b. $f(x) = \frac{1}{x}$

c. $f(x) = \sin x$

a. From the power rule, we know that $F(x) = Ax^{n+1}$ should be an antiderivative. In particular, $F'(x) = A(n+1)x^n = x^n$, so $A = \frac{1}{n+1}$. By Proposition **??**, the most general antiderivative of f is

$$F(x) = \frac{1}{n+1}x^{n+1}.$$

b. The power rule doesn't apply here (if it did, it would say that $\frac{d}{dx}[\frac{1}{1}x^0] = \frac{1}{x}$, but x^0 is a constant). Instead, we recall that the function with derivative $\frac{1}{x}$ is $\ln|x|$. So, by Proposition **??**, the most general antiderivative of f is

$$F(x) = \ln|x| + C.$$

c. Since sine and cosine alternate with their derivatives, we know that $F(x) = A\cos x$ for some constant A should be the antiderivative. Since $F'(x) = -A\sin x = \sin x$, we get that $A = -1$. So, by Proposition **??**, the most general antiderivative of f is

$$F(x) = -\cos x + C.$$

Playing this same game, we can complete the following useful table of antiderivatives:

Function	**Particular Antiderivative**	**Function**	**Particular Antiderivative**
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x)+g(x)$	$F(x)+G(x)$	$\cos x$	$\sin x$
x^n, $(n \neq 1)$	$\frac{1}{n+1}x^{n+1}$	$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\sec x \tan x$	$\sec x$
e^x	e^x	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
		$\frac{1}{1+x^2}$	$\tan^{-1} x$

Example 4.9.4. Find all functions g such that

$$g'(x) = 4\cos x + \frac{20x^4 - \sqrt{x^3}}{x}.$$

First we'll perform some algebraic manipulation to g' to get

$$g'(x) = 4\cos x + \frac{20x^4}{x} - \frac{\sqrt{x^3}}{x} = 4\cos x + 20x^3 - x^{1/2}.$$

Using the table rules above, we see that

$$g(x) = 4\sin x + 20\cdot\frac{1}{4}x^4 - \frac{1}{3/2}x^{3/2} + C = 4\sin x + 5x^4 - \frac{2}{3}x^{3/2} + C$$

is our most general form of the antiderivative and thus gives us all functions g.

Example 4.9.5. An object, initially at rest, falls off of a 200 foot building and constantly accelerates at $-32\,\text{ft/s}^2$. Find the equation of the position function.

We're given that the acceleration function for the object is

$$a(t) = -32.$$

This is the first derivative of the velocity function, so the antiderivative of $a(t)$ gets us

$$v(t) = -32t + C,$$

for some constant C. Since the object is initially at rest, $v(0) = 0$, so $C = 0$. Now, this function is the first derivative of the position function, so the antiderivative of $v(t)$ gets us

$$p(t) = -16t^2 + D,$$

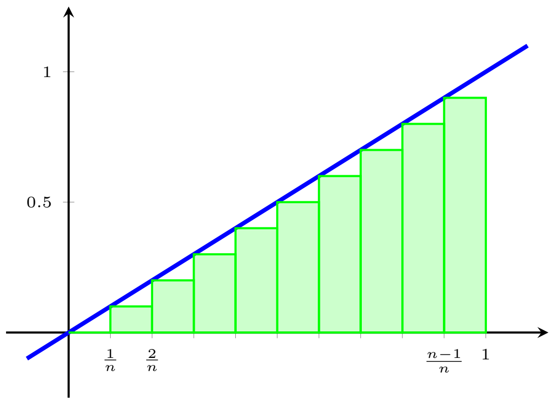
for some constant D. Since $p(0) = 200$, we deduce that $D = 200$. Thus, the precise position function for the object is $p(t) = -16t^2 + 200$, which exactly agrees with what we know about the kinematic motion equation from previous examples.

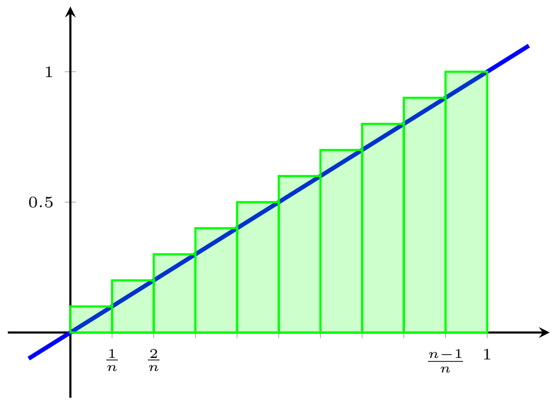
5 Integrals

5.1 Area and Distances

Example 5.1.1. We consider the graph of $y = x$ over the interval $[0, 1]$. Suppose we partition $[0, 1]$ into n-many smaller intervals. Let L_n be the *sum of the areas* of the rectangles in the picture on the left. Similarly, let R_n be the *sum of the areas* of the rectangles as shown in the picture on the right. Compute

$$\lim_{n\to\infty} L_n \qquad \text{and} \qquad \lim_{n\to\infty} R_n.$$





We see that

$$\begin{aligned} L_n &= \frac{1}{n}\left(\frac{0}{n}\right) + \frac{1}{n}\left(\frac{1}{n}\right) + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right) \\ &= \frac{1}{n^2}\left(0 + 1 + 2 + \cdots + n - 1\right) \\ &= \frac{1}{n^2} \cdot \frac{(n-1)n)}{2} \\ &= \frac{n^2 - n}{2n^2} \end{aligned}$$

We then have that

$$\lim_{n\to\infty} L_n = \frac{1}{2}$$

We see that

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right) + \frac{1}{n}\left(\frac{2}{n}\right) + \cdots + \frac{1}{n}\left(\frac{n}{n}\right) \\ &= \frac{1}{n^2}\left(1 + 2 + 3 + \cdots + n\right) \\ &= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \\ &= \frac{n^2 + n}{2n^2} \end{aligned}$$

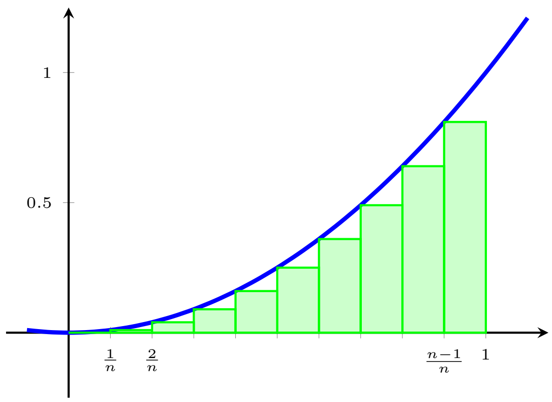
We then have that

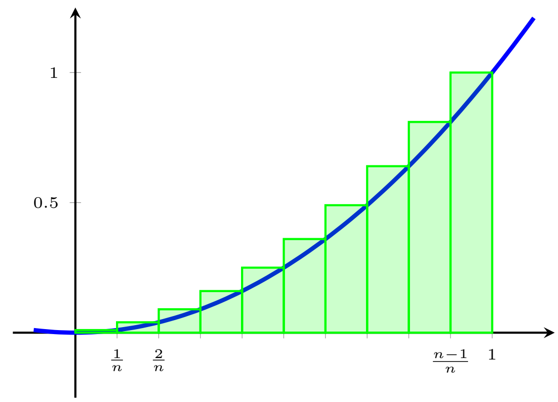
$$\lim_{n\to\infty} R_n = \frac{1}{2}$$

This limit makes sense. It's the area of a triangle with base 1 and height 1.

Example 5.1.2. Consider the graph of $y = x^2$ over the interval $[0, 1]$. Suppose we partition $[0, 1]$ into n-many smaller intervals. Let L_n be the *sum of the areas* of the rectangles as shown in the picture on the left. Let R_n be the *sum of the areas* of the rectangles as shown in the picture on the right. Compute

$$\lim_{n\to\infty} L_n \qquad \text{and} \qquad \lim_{n\to\infty} R_n.$$





We see that

$$\begin{aligned} L_n &= \frac{1}{n}\left(\frac{0}{n}\right)^2 + \frac{1}{n}\left(\frac{1}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2 \\ &= \frac{1}{n^3}\left(1^2 + \cdots + (n-1)^2\right) \\ &= \frac{1}{n^3} \cdot \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} \\ &\text{(sum of the first } n-1 \text{ squares)} \\ &= \frac{(n-1)(2n-1)}{6n^2} \\ &= \frac{2n^2 - 3n + 1}{6n^2}. \end{aligned}$$

We then have that

$$\lim_{n\to\infty} L_n = \frac{2}{6} = \frac{1}{3}.$$

We see that

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \\ &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &\text{(sum of the first } n \text{ squares)} \\ &= \frac{(n+1)(2n+1)}{6n^2} \\ &= \frac{2n^2 + 3n + 1}{6n^2}. \end{aligned}$$

We then have that

$$\lim_{n\to\infty} R_n = \frac{2}{6} = \frac{1}{3}.$$

It appears that as n increases, L_n and R_n both become better and better approximations of the area under the curve. So it seems like we should be able to define area A as a limit of the sums of the approximating rectangles. In other words,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n.$$

Then the region in the previous example has area $\frac{1}{3}$.

5.1.1 Area in General

Suppose we have a curve $y = f(x)$ on an interval $[a, b]$. Dividing it into n-many smaller intervals, we have that each interval has width

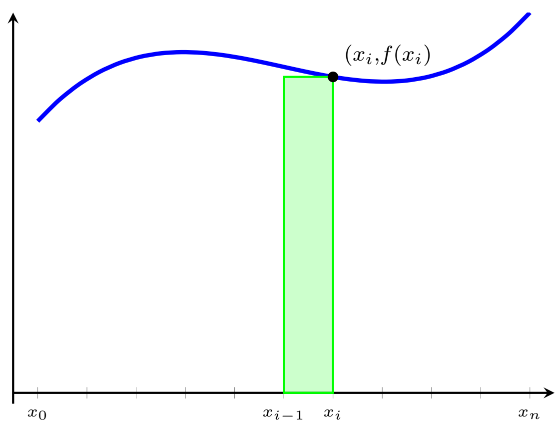
$$\Delta x = \frac{b-a}{n}.$$

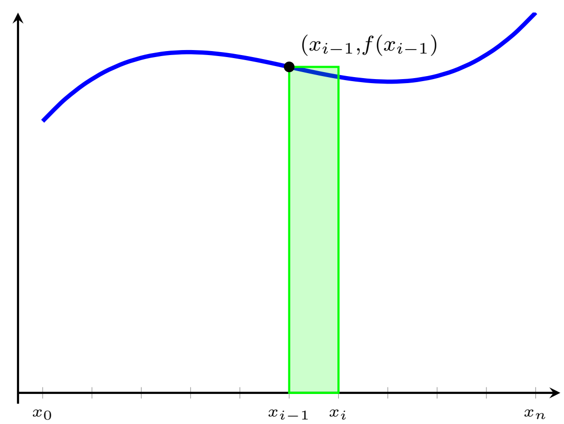
This means that we can divide $[a, b]$ into intervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n]$$

where $a = x_0$, $b = x_n$, and $x_i = a + i\Delta x$ for $i = 0, \ldots, n$. We then define R_n and L_n as follows:

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \qquad L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$





Definition. The **area** A of the region that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n.$$

Remark. In our definition, we focus on obtaining the area using only the left or only the right endpoint of each interval. In fact, it actually doesn't matter which point you choose in the interval - even arbitrary points in the interior will result in the same limit.

To avoid writing out all these terms, we use **sigma notation** to keep these things more compact.

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

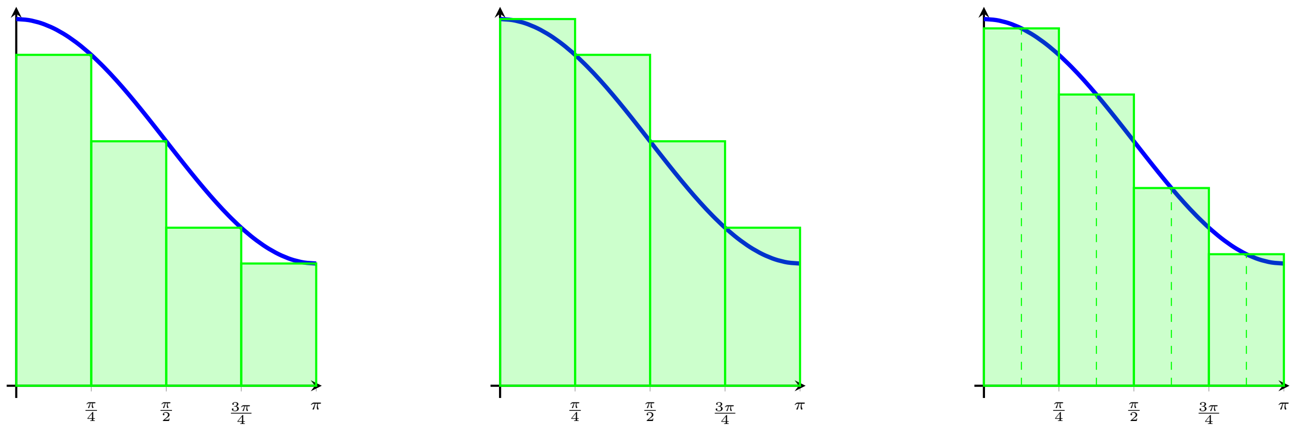
So, this means that we can rephrase the limit for area slightly as:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

Example 5.1.3. Estimate the area under the graph of $f(x) = 2 + \cos x$ from $x = 0$ to $x = \pi$ using 4 rectangles and

a. right endpoints. **b.** left endpoints. **c.** midpoints.



We note that $\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$, so $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, and $x_4 = \pi$.

1. For the right endpoints, we have

$$\begin{aligned} R_4 = \sum_{i=1}^{4} f(x_i)\Delta x &= \sum_{i=1}^{4} \left(2 + \cos\left(\frac{i\pi}{4}\right)\right) \cdot \frac{\pi}{4} \\ &= \frac{\pi}{4}\left(2 + \cos\frac{\pi}{4} + 2 + \cos\frac{\pi}{2} + 2 + \cos\frac{3\pi}{4} + 2 + \cos\pi\right) \\ &= \frac{7\pi}{4} \approx 5.4978. \end{aligned}$$

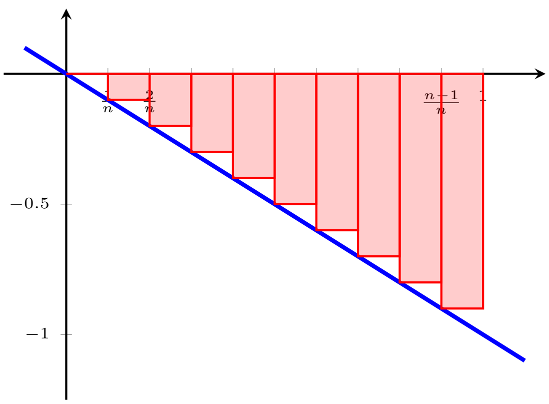
2. For the left endpoints, we have

$$\begin{aligned} L_4 = \sum_{i=1}^{4} f(x_{i-1})\Delta x &= \sum_{i=1}^{4} \left(2 + \cos\left(\frac{(i-1)\pi}{4}\right)\right) \cdot \frac{\pi}{4} \\ &= \frac{\pi}{4}\left(2 + \cos 0 + 2 + \cos\frac{\pi}{4} + 2 + \cos\frac{\pi}{2} + 2 + \cos\frac{3\pi}{4}\right) \\ &= \frac{9\pi}{4} \approx 7.0686. \end{aligned}$$

3. For the midpoints, we have

$$\begin{aligned} M_4 = \sum_{i=1}^{4} f\left(\frac{x_i - x_{i-1}}{2}\right)\Delta x &= \sum_{i=1}^{4} \left(2 + \cos\left(\frac{(2i-1)\pi}{8}\right)\right) \cdot \frac{\pi}{4} \\ &= \frac{\pi}{4}\left(2 + \cos\frac{\pi}{8} + 2 + \cos\frac{3\pi}{8} + 2 + \cos\frac{5\pi}{8} + 2 + \cos\frac{7\pi}{8}\right) \\ &= 2\pi \approx 6.2832. \end{aligned}$$

Example 5.1.4. Consider now the graph of $f(x) = -x$ over the interval $[0, 1]$. And we want to compute the area between the x-axis and the curve $y = f(x)$. We set up the sums R_n and L_n as before and try to compute the area.

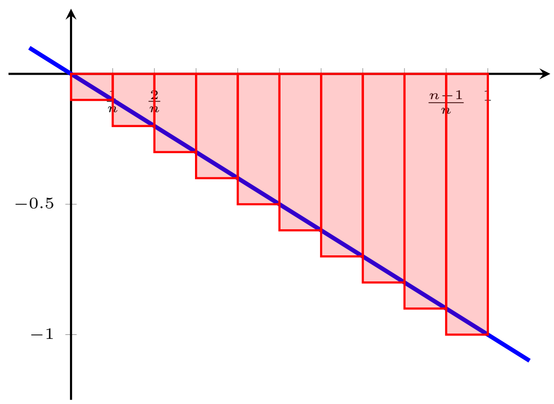


From the general area setup, $a = 0$, $b = 1$. We're using n rectangles. Thus $\Delta x = \frac{1}{n}$ and $x_i = 0 + i\Delta x = \frac{i}{n}$.

$$L_n = \sum_{i=0}^{n-1} \Delta x \cdot f(x_i) = \sum_{i=0}^{n-1} \frac{1}{n}\left(-\frac{i}{n}\right)$$
$$= -\frac{1}{n^2}\sum_{i=0}^{n-1} i$$
$$= -\frac{1}{n^2} \cdot (0 + 1 + \cdots + n - 1)$$
$$= -\frac{1}{n^2} \cdot \frac{(n-1)n}{2}$$

We then have that

$$\lim_{n\to\infty} L_n = -\frac{1}{2}$$



From the general area setup, $a = 0$, $b = 1$. We're using n rectangles. Thus $\Delta x = \frac{1}{n}$ and $x_i = 0 + i\Delta x = \frac{i}{n}$.

$$R_n = \sum_{i=1}^{n} \Delta x \cdot f(x_i) = \sum_{i=1}^{n} \frac{1}{n}\left(-\frac{i}{n}\right)$$
$$= -\frac{1}{n^2}\sum_{i=1}^{n} i$$
$$= -\frac{1}{n^2} \cdot (1 + \cdots + n)$$
$$= -\frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

We then have that

$$\lim_{n\to\infty} R_n = -\frac{1}{2}$$

What this tells us is that our procedure for finding the area actually finds the *signed area*. When $f(x) > 0$ on some interval, then the area will be positive. When $f(x) < 0$ on some interval, then the area will be negative. If $f(x)$ changes sign on some interval, then the area formula computes the *net area.*

Example 5.1.5. Set up a sum representing an approximation of the area under the curve $y = f(x)$ on the interval $[a, b]$ using n rectangles:

a. $f(x) = \dfrac{1}{(1+x)^2}$, $[0, 1]$

$$\sum_{i=1}^{n} \frac{1}{(1+i/n)^2} \cdot \frac{1}{n}$$

b. $f(x) = \sqrt{1+x}$, $[2, 4]$

$$\sum_{i=1}^{n} \frac{1}{(1+2+2i/n)^2} \cdot \frac{2}{n}$$

c. $f(x) = x^x$, $[1, 11]$

$$\sum_{i=1}^{n} (1+10i/n)^{(1+10i/n)} \cdot \frac{10}{n}$$

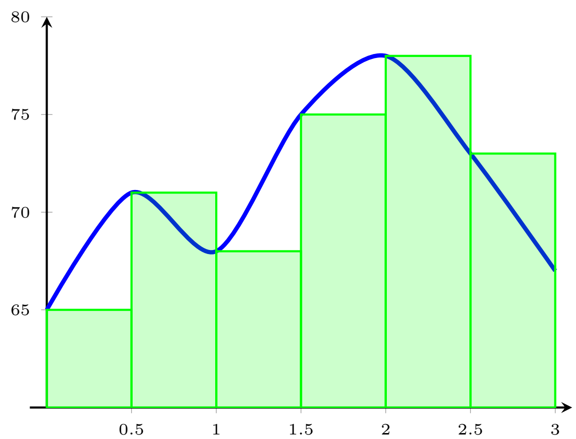
5.1.2 The Distance Problem

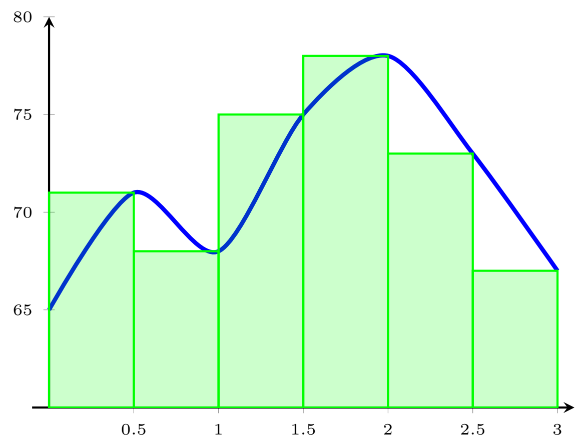
Example 5.1.6. Suppose the odometer in the car is broken and we want to figure out the distance we have traveled over a 3-hour time period. Since

$$\text{distance} = \text{velocity} \times \text{time},$$

we could take spedometer readings every half hour and use the above formula to estimate the distance traveled.

Time (h)	0	0.5	1	1.5	2	2.5	3
Velocity (mi/h)	65	71	68	75	78	73	67



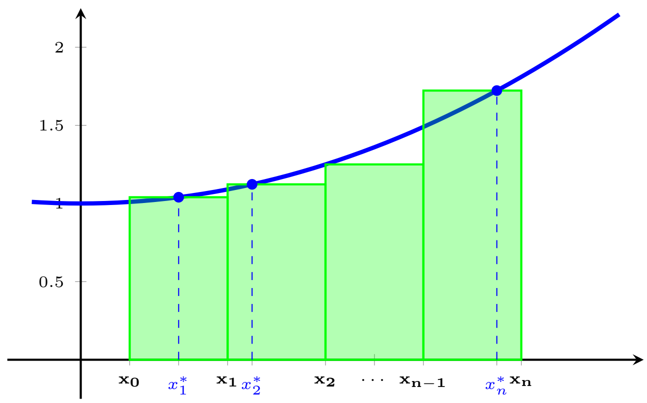


$$\begin{aligned}\text{dist} \approx L_6 &= \frac{1}{2}(65+71+68+75+78+73)\\ &= \frac{1}{2}(430)\\ &= 215\,\text{mi.}\end{aligned}$$

$$\begin{aligned}\text{dist} \approx R_6 &= \frac{1}{2}(65+71+68+75+78+73)\\ &= \frac{1}{2}(432)\\ &= 216\,\text{mi.}\end{aligned}$$

5.2 The Definite Integral

As we talked about before, we chose right and left endpoints or midpoints for approximating the area under the curve, but it ultimately didn't even matter - we could base our rectangles at a height chosen by any number x_i^* chosen in the evenly-spaced interval $[x_{i-1}, x_i]$. (In fact, as it turns out, we don't even have to evenly space our intervals, just as long as the length of each subinterval tends to 0, we should end up with the same area; but we won't stress this fact.)



Definition. The **Riemann sum** is the sum of the rectangles approximating the area under a curve,

$$\sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where x_i^* is any point in the interval $[x_{i-1}, x_i]$. If we choose x_i^* to always be the left (resp. right) endpoint, we call this a **left** (resp. **right**) **Riemann sum**.

Definition. If f is defined on $[a, b]$, then the **definite integral of f from a to b** is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

If the limit exists, we say that f is **integrable on** $[\mathbf{a}, \mathbf{b}]$. Here $\int$ is called the **integral sign**, f is called the **integrand**, and a and b are called the **limits of integration** (in particular, a is called the **lower limit** and b is called the **upper limit**). As before, dx is just a differential but it doesn't have much meaning by itself – think of it as a bookend for the integral notation.

Proposition 5.2.1. *If f is continuous or has only finitely many holes/jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$, that is $\int_a^b f(x)\,dx$ exists.*

This is great, because it says that it's easier to be integrable than it is to be differentiable. Moreover, almost every function we've worked with in this class satisfies these properties. The downside is that we allow functions that are less well-behaved, and so computing integrals can require a bit more effort; such things will be reserved for a future course.

Example 5.2.2. Express the following as a definite integral on the given interval:

a. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{3}{x_i}\right) \Delta x$ on $[1, 5]$

b. $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{4i}{n}} \cdot \frac{4}{n}$ on $[0, 4]$

a. $\int_1^5 \left(2 + \frac{3}{x}\right) dx$

b. $\int_0^4 \sqrt{x}\, dx$

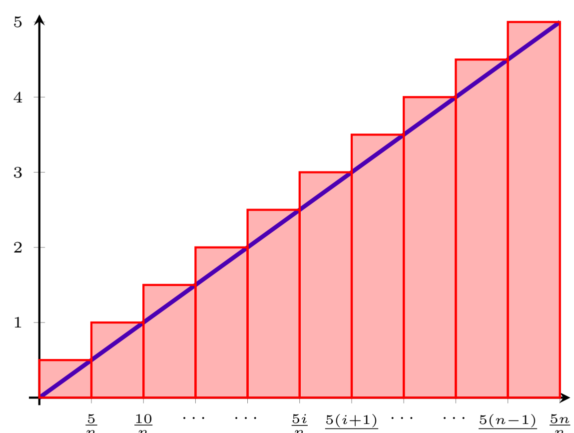
Example 5.2.3. Express the following as a definite integral on an interval where the left endpoint is 2.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \cos(x_i)\, \frac{3}{n}.$$

$$\int_2^5 \cos(x)\, dx$$

Example 5.2.4. Evaluate the following definite integral.

$$\int_0^5 x\, dx$$



Here $f(x) = x$, $\Delta x = \frac{5-0}{n} = \frac{5}{n}$, $x_0 = 0$, and $x_i = x_0 + i \cdot \Delta x = \frac{5i}{n}$. So,

$$\int_0^5 x\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5i}{n}\right)\left(\frac{5}{n}\right) = \lim_{n\to\infty} \frac{25}{n^2} \cdot \left(\sum_{i=1}^{n} i\right) = \lim_{n\to\infty} \frac{25n^2 + 25n}{2n^2} = \frac{25}{2}.$$

And indeed, $\frac{25}{2}$ is the area we got by just knowing about the area of a triangle.

Proposition 5.2.5 (Properties of Definite Integrals)**.** *Let $f(x)$, $g(x)$ be integrable functions on $[a, b]$ and c a real number. Then*

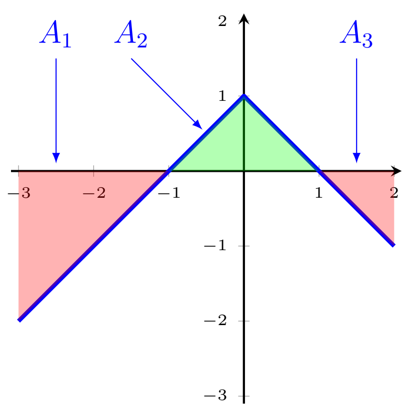
1. $\int_a^b c\,dx = c(b-a)$
2. $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$
3. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Proof. The proof of each property follows easily from properties of summations and the algebra of limits. Only the last one seems kind of strange, but indeed it comes from the fact that when looking at $\int_a^b$, we have $\Delta x = \frac{b-a}{n}$ and when looking at $\int_b^a$, we have $\Delta x = \frac{a-b}{n} = -\frac{b-a}{n}$. $\square$

When $f(x)$ is positive on $[a, b]$, the definite integral $\int_a^b f(x)\,dx$ represents the area above the x-axis and under the curve $y = f(x)$. When $f(x)$ takes both positive and negative values on $[a, b]$, the definite integral $\int_a^b f(x)\,dx$ represents the net area (that is, the area above x-axis and below the curve $y = f(x)$, minus the area below the x-axis and above the curve $y = f(x)$).

Example 5.2.6. Evaluate the following integrals by interpreting it in terms of area: $\int_{-3}^{2} (1 - |x|)\,dx$

Drawing this out, we see that we just have the area of green triangle minus the area of the red triangles.



So,

$$\int_{-3}^{2} (1 - |x|)\,dx = A_1 + A_2 + A_3 = -\frac{1}{2}(2)(2) + \frac{1}{2}(2)(1) - \frac{1}{2}(1)(1) = -\frac{3}{2}.$$

This next result tells us how we can combine integrals on adjacent intervals.

Proposition 5.2.7. *Suppose $f(x)$ is integrable on $[a,c]$ and $a \le b \le c$. Then certainly f is integrable on both $[a,b]$ and $[b,c]$ and*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^b f(x)\,dx.$$

Example 5.2.8. Suppose $\int_0^{27} f(x)\,dx = 10$ and $\int_0^{15} f(x)\,dx = 3$. Find $\int_{15}^{27} f(x)\,dx$.

By Proposition 5.2.7,

$$\begin{aligned}\int_0^{15} f(x)\,dx + \int_{15}^{27} f(x)\,dx &= \int_0^{27} f(x)\,dx \\ 3 + \int_{15}^{27} f(x)\,dx &= 10 \\ \int_{15}^{27} f(x)\,dx &= 7\end{aligned}$$

Example 5.2.9. Suppose $\int_0^4 f(x)\,dx = 2$, $\int_4^6 f(x)\,dx = 3$, and $\int_0^6 g(x)\,dx = 9$, find

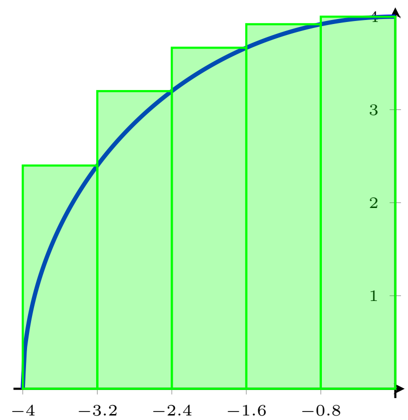
$$\int_0^6 3f(x) - g(x)\,dx.$$

$$\begin{aligned}\int_0^6 \left(3f(x) - g(x) + 1\right)\,dx &= \int_0^6 3f(x)\,dx - \int_0^6 g(x)\,dx + \int_0^6 1\,dx \\ &= 3\int_0^6 f(x)\,dx - \int_0^6 g(x)\,dx + \int_0^6 1\,dx \\ &= 3\left(\int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx\right) - \int_0^6 g(x)\,dx + \int_0^6 1\,dx \\ &= 3(2+3) - 9 + 6 = 12.\end{aligned}$$

Example 5.2.10. Approximate the following definite integral using a right Riemann sum with 5 rectangles: $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$.

We have here that $f(x) = \sqrt{16 - x^2}$, $\Delta x = \frac{0-(-4)}{5} = 0.8$, and

$$x_0 = -4, \quad x_1 = -3.2, \quad x_2 = -2.4, \quad x_3 = -1.6, \quad x_4 = -0.8, \quad x_5 = 0.$$

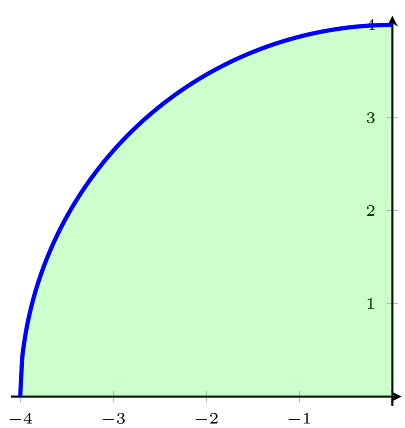


So, the sum of the areas of these rectangles is

$$\begin{aligned}\sum_{i=1}^{5} f(x_i)\,\Delta x &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x \\ &\approx (2.4)\,0.8 + (3.2)\,0.8 + (3.3667)\,0.8 + (3.919)\,0.8 + (4)\,0.8+ \\ &\approx 13.748\end{aligned}$$

Example 5.2.11. Evaluate the definite integral in Example 5.2.10 by interpreting it in terms of area.

Notice that, if $y = \sqrt{16 - x^2}$, then $y^2 = 16 - x^2$, and thus $x^2 + y^2 = 16$, so the graph traces out the top half of the circle of radius 4. Drawing this out, we see that we're just looking at the area under one quarter of this circle.



With this in mind, the area is quite simply

$$\int_{-4}^{0} \sqrt{16 - x^2}\, dx = \frac{1}{4}\pi(4)^2 = 4\pi.$$

Since Riemann sums can be very hard to get a hold of explicitly and certain curves don't bound areas that are easily computed via geometric means, it's at least useful to be able to bound the value of a definite integral.

Proposition 5.2.12. *Let f, g be integrable on $[a, b]$ and let m, M be constants.*

a. *If $f(x) \geq 0$, then*

$$\int_a^b f(x)\,dx \geq 0.$$

b. *If $f(x) \geq g(x)$, then*

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

c. *If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then*

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a)$$

Proof. For part (a), since $f(x) \geq 0$, in the Riemann sum we must have that for each x_i^* in the partition, the corresponding rectangle has area satisfying $f(x_i^*)\Delta x \geq 0$. Applying the limit inequality in Lemma 2.3.9 to the sum completes the result. Parts (b) and (c) are nearly identical arguments as well. □

Example 5.2.13. Use the Proposition 5.2.12 to estimate (i.e. find bounds) for the value of $\displaystyle\int_{\pi/4}^{\pi/3} \tan(x)\,dx$.

Since $\tan(x)$ is increasing on the interval $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, we could apply part 3 of Proposition 5.2.12 using $m = \tan\left(\frac{\pi}{4}\right) = 1$ and $m = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ as upper bounds. Hence

$$0.262 \approx \frac{\pi}{12} = \int_{\pi/4}^{\pi/3} 1\,dx \leq \int_{\pi/4}^{\pi/3} \tan(x)\,dx \leq \int_{\pi/4}^{\pi/3} \sqrt{3}\,dx = \frac{\pi\sqrt{3}}{12} \approx 0.453$$

Going a bit further, we could also use the fact that $\tan(x)$ is concave up in this interval and take the line between $\left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{\pi}{3}, \sqrt{3}\right)$ as an upper bound and apply part 2 of Proposition 5.2.12; explicitly, this is the line given by $y = \frac{12(\sqrt{3}-1)}{\pi}\left(x - \frac{\pi}{4}\right) + 1$. Hence

$$0.262 \approx \frac{\pi}{12} = \int_{\pi/4}^{\pi/3} 1\,dx \leq \int_{\pi/4}^{\pi/3} \tan(x)\,dx \leq \int_{\pi/4}^{\pi/3} \frac{12(\sqrt{3}-1)}{\pi}\left(x - \frac{\pi}{4}\right) + 1\,dx = \frac{\pi}{24}(1+\sqrt{3}) \approx 0.357.$$

Any estimate in this range is actually pretty good. The actual area is

$$\frac{\ln(2)}{2} \approx 0.346$$

5.3 The Fundamental Theorem of Calculus

Imagine that we have a curve $y = f(x)$ and we want to know how the area is changing if we fix the left endpoint and let the right endpoint vary. As a function (an "accumulating function"), we write

$$A(x) = \int_a^x f(t)\,dt$$

Visually, Figure 5.3.1 shows how the area is "accumulating" under the curve as we increase x. See https://www.desmos.com/calculator/cawzpbcwwr for an interactive Desmos graph of this accumulation function.

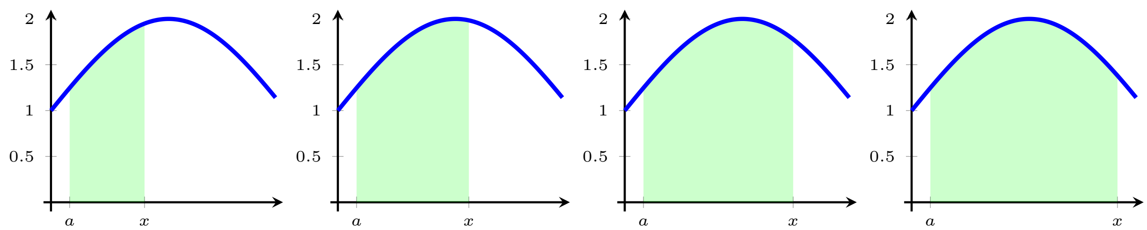
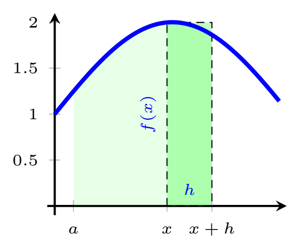


Figure 5.3.1: Area accumulating under the curve as x increases.

Example 5.3.1. Let $f(t) = 2t$, $A(x) = \int_0^x f(t)\,dt$. Find $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, and conjecture about $A(x)$.

We see that $A(0) = 0$, $A(1) = 1$, $A(2) = 4$, $A(3) = 9$, $A(4) = 16$. It seems that $A(x) = x^2$. This is interesting because $A(x)$ looks a lot like an antiderivative for f.



Notice that the difference in areas $A(x+h) - A(x)$ is approximately the area of the rectangle $h \cdot f(x)$. Well, this rearranges to

$$f(x) \approx \frac{A(x+h) - A(x)}{h}.$$

Well, as h gets smaller, the approximation gets better and better, so in fact,

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = A'(x).$$

This gives us the following

Theorem 5.3.2 (Fundamental Theorem of Calculus, pt. I)**.** *If f is continuous on $[a, b]$, then the function A defined by*

$$A(x) = \int_a^x f(t)\,dt, \quad a \le x \le b,$$

is continuous on $[a, b]$, differentiable on (a, b), and $A'(x) = f(x)$.

Example 5.3.3. Find $A'(x)$ given $A(x) = \int_0^x \frac{t^3+t-1}{\sqrt{t^2+7}}\, dt$.

By the Fundamental Theorem of Calculus (FTC),

$$A'(x) = \frac{x^3+x-1}{\sqrt{x^2+7}}.$$

Exercise 5.3.1. Find $\frac{d}{dx}[A(x)]$ where $A(x) = \int_0^x \arctan(t)\, dt$.

Example 5.3.4. Find $\frac{d}{dx}\int_0^{x^3} \arctan t\, dt$.

Notice that, given $A(x)$ as in the previous exercise, this time we're looking for $\frac{d}{dx}[A(x^3)]$. This is going to require a chain rule. Thus we have

$$\frac{d}{dx}\left[A(x^3)\right] = A'(x^3)\cdot\frac{d}{dx}\left[x^3\right] = \arctan\big(x^3\big)\cdot 3x^2.$$

Exercise 5.3.2. Find $\frac{d}{dx}\int_x^8 t^7\, dt$.

In order to apply FTC, we need to have the lower limit fixed and the upper limit to be a function of x. Since

$$\int_x^8 t^7\, dt = -\int_8^x t^7\, dt,$$

then we have that

$$\frac{d}{dx}\int_x^8 t^7\, dt = -\frac{d}{dx}\int_8^x t^7\, dt = -x^7$$

Of course, having a negative output on our accumulation function here makes sense. As shown in Figure 5.3.2, when the left endpoint is variable and the right endpoint is fixed, the accumulated area is *decreasing* as x *increases*.

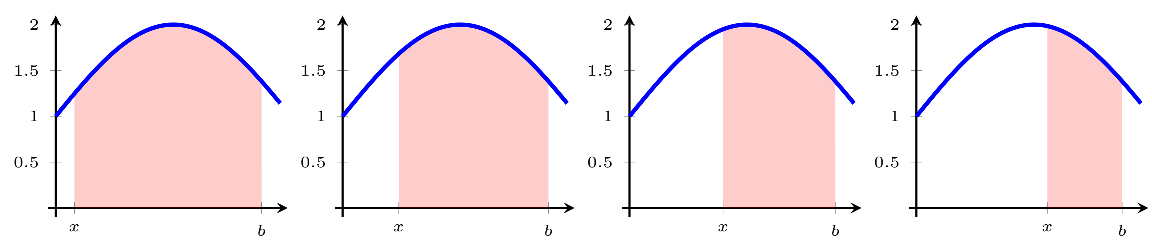


Figure 5.3.2: With the right endpoint fixed, accumulated area decreases as x increases.

Exercise 5.3.3. Find $\frac{d}{dx}\int_{\sin x}^{x^2+1} \sqrt{t+1}\, dt$. *[Hint: use Proposition 5.2.7]*

If f is continuous on $[a, b]$, then we saw that the accumulation function A was an antiderivative for f. This means that any antiderivative F of f on $[a, b]$ should be of the form $F(x) = A(x) + C$, where C is a constant. Then

$$F(b) - F(a) = A(b) + C - (A(a) + C) = A(b) - A(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt$$

where the integral $\int_a^a f(t)\,dt = 0$ because there is no area comprised by a vertical line segment at a. This gives us, now, an easy way to compute definite integrals and again highlights the interplay between antiderivatives and definite integrals.

Theorem 5.3.5 (Fundamental Theorem of Calculus, pt. II)**.** *If F is continuous on the interval $[a, b]$ and F is an antiderivative of f, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Remark. In lieu of the following theorem, given an antiderivative F for f, the following notations are convenient ways to represent $\int_a^b f(x)\,dx$:

$$[F(x)]_a^b \qquad \text{or} \qquad F(x)\big|_a^b$$

Example 5.3.6. Find the exact value of $\displaystyle\int_0^4 x\,dx$.

Recall that the general antiderivative of x is $\frac{1}{2}x^2 + C$. By the FTC we have

$$\int_0^1 x\,dx = \left[\frac{1}{2}(4)^2 + C\right] - \left[\frac{1}{2}(0)^2 + C\right] = 8.$$

Indeed, this agrees with what we know about the area of a triangle of base length 4 and height 4.

Example 5.3.7. Find the exact value of $\displaystyle\int_0^1 x^2\,dx$.

Recall that the general antiderivative of x^2 is $\frac{1}{3}x^3 + C$. By the FTC we have that

$$\int_0^1 x^2\,dx = \left[\frac{1}{3}(1)^3 + C\right] + \left[\frac{1}{3}(0)^3 + C\right] = \frac{1}{3}.$$

This is exactly what we saw in a previous example.

These example also highlight a useful fact about using the Fundamental Theorem to compute definite integrals - we don't need the general form of the antiderivative as the added constant "$+C$" will cancel itself out in the end, so any antiderivative will do. As such, we might as well take $C = 0$.

Example 5.3.8. Find the area under the curve $y = \cos x$ over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Recall that $y = \sin x$ is an antiderivative for $\cos x$. Thus, by the Evaluation Theorem,

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) = 1 - (-1) = 2.$$

5.4 Indefinite Integrals and the Net Change Theorem

5.4.1 Indefinite Integrals

Because of the relationship between the antiderivative and integral, we use the notation $\int f(x)\,dx$ for the general antiderivative of f, and we call this the **indefinite integral**. Specifically,

$$\int f(x)\,dx = F(x) \quad \text{is equivalent to} \quad F'(x) = f(x).$$

Remark. The definite integral $\int_a^b f(x)\,dx$ is a *number* and the indefinite integral $\int f(x)\,dx$ is a *function* (or a family of functions).

Example 5.4.1. Evaluate the following indefinite integral: $\displaystyle\int \left(14x^6 - \sec x \tan x\right)\,dx.$

Recall that an antiderivative for x^6 is $\frac{1}{7}x^7$ and the general antiderivative of $\sec x \tan x$ is $\sec x$. So, applying our properties of integrals,

$$\begin{aligned}
\int \left(14x^6 - \sec x \tan x\right)\,dx &= \int 14x^6\,dx - \int \sec x \tan x\,dx \\
&= 14\int x^6\,dx - \int \sec x \tan x\,dx \\
&= 14\left(\frac{1}{7}x^7\right) - (\sec x) + C \\
&= 2x^7 - \sec x + C.
\end{aligned}$$

Example 5.4.2. Evaluate the following indefinite integral: $\displaystyle\int \frac{\sqrt{t}-1}{\sqrt{t}}\,dt.$

Recall that an antiderivative for 1 is t and an antiderivative for $t^{-1/2}$ is $-2t^{1/2}$. So, simplifying the integrand and applying properties of integrals, we have

$$\begin{aligned}
\int \frac{\sqrt{t}-1}{\sqrt{t}}\,dt &= \int \left(1 - \frac{1}{\sqrt{t}}\right)\,dt \\
&= \int \left(1 - t^{-1/2}\right)\,dt \\
&= \int 1\,dt - \int t^{-1/2}\,dt \\
&= t - (-2t^{1/2}) + C \\
&= t + 2\sqrt{t} + C.
\end{aligned}$$

Example 5.4.3. Evaluate the following indefinite integral: $\displaystyle\int 5^w \ln(5)\, dw$

$$\int 5^w \ln(5)\, dw = 5^w + C.$$

Example 5.4.4. Evaluate the following indefinite integral: $\displaystyle\int \cos^2\theta + \sin^2\theta\, d\theta$

Since $\cos^2\theta + \sin^2\theta = 1$, we have

$$\int \cos^2\theta + \sin^2\theta\, d\theta = \int 1\, d\theta = \theta + C.$$

Example 5.4.5. Evaluate the following indefinite integral: $\displaystyle\int \sin^2\left(\frac{x}{2}\right)\, dx$

Hint: $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$

Since $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$, then

$$\begin{aligned}\int \sin^2\left(\frac{x}{2}\right)\, dx &= \int \frac{1}{2}\left(1 - \cos(x)\right)\, dx \\ &= \frac{1}{2}\int 1 - \cos(x)\, dx \\ &= \frac{1}{2}\left(x - \sin(x)\right) + C.\end{aligned}$$

5.4.2 Applications

Since position is the antiderivative of velocity, then for a particle with a given velocity function $v(t)$ and position function $p(t)$, we have that

$$\int_a^b v(t)\,dt = p(t_2) - p(t_1)$$

and so the definite integral of the velocity function computes the **displacement of an object from time $t = a$ to time $t = b$**. If $v(t)$ is always positive, then this displacement is exactly the same as the distance traveled, and so the **total distance traveled from time $t = a$ to time $t = b$** is exactly

$$\int_a^b |v(t)|\,dt.$$

In practice, this means that we'll have to split up the integral into pieces where the velocity is positive.

Example 5.4.6. Suppose $v(t)$ is positive on the time intervals $[t_1, t_2]$ and $[t_3, t_4]$ and is negative on the time interval $[t_2, t_3]$. Then the total distance traveled from time t_1 to time t_4 is given by

$$\begin{aligned}\int_{t_1}^{t_4} |v(t)|\,dt &= \int_{t_1}^{t_2} |v(t)|\,dt + \int_{t_2}^{t_3} |v(t)|\,dt + \int_{t_3}^{t_4} |v(t)|\,dt \\ &= \int_{t_1}^{t_2} v(t)\,dt + \int_{t_2}^{t_3} -v(t)\,dt + \int_{t_3}^{t_4} v(t)\,dt.\end{aligned}$$

Example 5.4.7. Suppose a particle is moving along a line with velocity function $v(t) = t^2 - 5t + 6$. Find both the displacement and distance traveled by the particle during the time interval $[1, 5]$.

$$\begin{aligned}\text{displacement} &= \int_2^5 \left(t^2 - 5t + 6\right)\,dt \\ &= \left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t\right)\Big|_2^5 \\ &= \left(\frac{1}{3}(5)^3 - \frac{5}{2}(5)^2 + 6(5)\right) - \left(\frac{1}{3}(2)^3 - \frac{5}{2}(2)^2 + 6(2)\right) \\ &= \frac{16}{3}.\end{aligned}$$

Now, notice that the velocity is negative on $(2, 3)$, so the particle is back-tracking. This means that there is some distance that the particle travels in one direction that cancels out with some of the distance traveled in the opposite direction. To get the total distance traveled, we'll need to handle the negative velocity cases separately from the positive velocities. In particular, the total distance traveled is

$$\begin{aligned}\int_1^5 \left|t^2 - 5t + 6\right|\,dt &= \int_1^2 t^2 - 5t + 6\,dt + \int_2^3 -\left(t^2 - 5t + 6\right)\,dt + \int_3^5 t^2 - 5t + 6\,dt \\ &= \frac{5}{6} + \frac{1}{6} + \frac{14}{3} \\ &= \frac{17}{3}.\end{aligned}$$

5.5 The Substitution Rule

Example 5.5.1 (Warm Up)**.** Find $\frac{d}{dx}\left[(3x^2-5)^8\right]$.

Using the chain rule, we have

$$\frac{d}{dx}\left[\left(3x^2-5\right)^8\right] = 8\left(3x^2-5\right)^7 \cdot 6x.$$

Recall that the chain rule says

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

where $g(x)$ is your "inner function" and $f(x)$ is your "outer function". Recall also that, if $u = g(x)$ is a differentiable function, then in the language of differentials, we have $du = g'(x)\,dx$.

The following rule combines these two concepts in a way that is exactly analogous to the chain rule for differentiation.

Proposition 5.5.2 (Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I, and if f is continuous on I, then*

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

Example 5.5.3. Using the substitution rule, evaluate $\int 8\left(3x^2-5\right)^7 \cdot 6x\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$u = g(x) = 3x^2 - 5.$$
$$du = g'(x)\,dx = 6x\,dx.$$

Hence, by the substitution rule,

$$\begin{aligned}\int 8\left(3x^2-5\right)^7 \cdot 6x\,dx &= \int 8u^7\,du \\ &= u^8 + C \\ &= \left(3x^2-5\right)^8 + C \qquad \text{(substitute back in for } u\text{).}\end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.4. Using the substitution rule, evaluate $\int e^{-2x}\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$\begin{aligned} u &= g(x) = -2x \\ du &= g'(x)\,dx = -2\,dx \quad \Rightarrow \quad -\frac{1}{2}\,du = dx. \end{aligned}$$

Hence, by the substitution rule,

$$\begin{aligned} \int e^{-2x}\,dx &= \int e^{u}\left(-\frac{1}{2}\right)\,du \\ &= -\frac{1}{2}\int e^{u}\,du \\ &= -\frac{1}{2}e^{u} + C \\ &= -\frac{1}{2}e^{-2x} + C \qquad \text{(substitute back in for } u\text{).} \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.5. Using the substitution rule, evaluate $\int \frac{(\ln x)^2}{x}\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$\begin{aligned} u &= g(x) = \ln x \\ du &= g'(x)\,dx = \frac{1}{x}\,dx. \end{aligned}$$

Hence, by the substitution rule,

$$\begin{aligned} \int \frac{(\ln x)^2}{x}\,dx &= \int u^2\,du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(\ln x)^3 + C \qquad \text{(substitute back in for } u\text{).} \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.6. Using the substitution rule, evaluate $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$u = g(x) = x^2 + 1 \tag{5.5.1}$$

$$du = g'(x)\,dx = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$$

This gives us

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{x^2}{\sqrt{u}}\left(\frac{1}{2}\right)\,du$$

But what do we do with the x^2 term? Well notice that we can rearrange Equation 5.5.1 to get $x^2 = u - 1$, so

$$\begin{aligned}
&= \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du \\
&= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)\,du \\
&= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\
&= \frac{1}{3}u^{3/2} - u^{1/2} + C \\
&= \frac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C \qquad \text{(substitute back in for } u\text{)}.
\end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.7. Using the substitution rule, evaluate $\int \tan(x)\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function". But where can this come from? First we recall that $\tan x = \frac{\sin x}{\cos x}$ and let

$$u = g(x) = \cos x$$

$$du = g'(x)\,dx = -\sin x\,dx \quad \Rightarrow \quad -du = \sin x\,dx$$

Hence, by the substitution rule,

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\
&= -\int \frac{du}{u} \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C \qquad \text{(substitute back in for } u\text{)}.
\end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Proposition 5.5.8 (Substituion Rule for Definite Integrals). *If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof. Let F be an antiderivative for f. Then $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ by the substitution rule. So, we have

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = F(g(x))\big|_a^b = F(g(b)) - F(g(a))$$

and

$$\int_{g(a)}^{g(b)} f(u)\, du = F(u)\big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)),$$

whence the definite integrals must be equal. □

Example 5.5.9. Evaluate $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

Let

$$u = g(x) = \frac{1}{x}$$
$$du = g'(x)\, dx = -\frac{1}{x^2}\, dx \quad \Rightarrow \quad -du = \frac{1}{x^2}\, dx.$$

Our new endpoints then become

$$u(1) = g(1) = 1$$
$$u(2) = g(2) = \frac{1}{2}.$$

Thus, applying the substitution rule, we have

$$\begin{aligned}\int_{x=1}^{x=2} \frac{e^{1/x}}{x^2}\, dx &= \int_{u=1}^{u=1/2} e^u(-du)\\ &= -\int_1^{1/2} e^u\, du\\ &= \int_{1/2}^1 e^u\, du\\ &= e^u\big|_{1/2}^1\\ &= e^1 - e^{1/2} = e - \sqrt{e}.\end{aligned}$$

Example 5.5.10. Evaluate $\displaystyle\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

Let

$$u = \arcsin x$$
$$du = \frac{1}{\sqrt{1-x^2}}\,dx.$$

Our new endpoints then become

$$u(0) = \arcsin(0) = 0$$
$$u\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

Thus, applying the substitution rule, we have

$$\begin{aligned}\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx &= \int_0^{\pi/6} u\,du \\ &= \frac{1}{2}u^2\Big|_0^{\pi/6} \\ &= \frac{1}{2}\left(\frac{\pi}{6}\right)^2 - \frac{1}{2}(0)^2 = \frac{\pi^2}{72}.\end{aligned}$$

Example 5.5.11. Given

$$\int \arctan(u)\,du = u\arctan(u) - \frac{1}{2}\ln\bigl(1+u^2\bigr) + C,$$

evaluate the indefinite integral

$$\int \arctan(2x^3)\cdot 6x^2\,dx.$$

Making the substitution

$$u = 2x^3$$
$$du = 6x^2$$

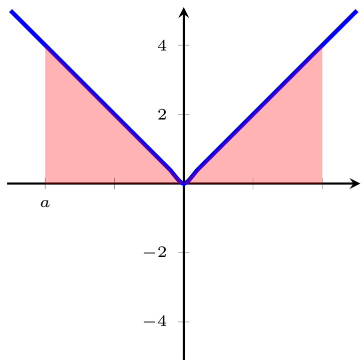
we get

$$\begin{aligned}\int \arctan(2x^3)\cdot 6x^2\,dx &= \int \arctan(u)\,du \\ &= u\arctan(u) - \frac{1}{2a}\ln\bigl(1+a^2u^2\bigr) + C \\ &= 2x^3\arctan\bigl(2x^3\bigr) - \frac{1}{2}\ln\bigl(1+4x^6\bigr) + C\end{aligned}$$

Exercise 5.5.1. Evaluate $\displaystyle\int_0^{\pi/4} \sec^4\theta\,d\theta$. *[Hint:* $\sec^2\theta = 1+\tan^2\theta$*]*

Recall that a function f is **even** if $f(-x) = f(x)$, and f is **odd** if $f(-x) = -f(x)$, where x is any real number in the domain. The following result tells us about the symmetry of these functions as they relate to definite integrals.

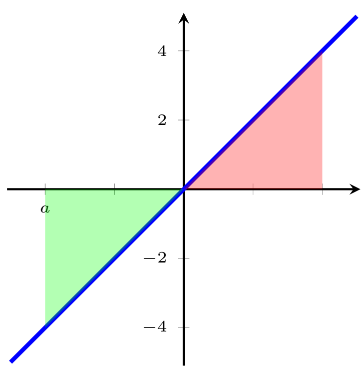
Example 5.5.12. One simple example of an even function is $f(x) = |x|$. Notice that, for some positive real number a, the integral $\int_{-a}^{a} f(x)\,dx$ is represented by the picture below.



Notice that the shaded regions to the left and right of the y-axis are equal, so the area under the curve $y = f(x)$ over the interval $[-a, a]$ is double the area found over the interval $[0, a]$. In other words,

$$\int_{-a}^{a} |x|\,dx = 2\int_{0}^{a} |x|\,dx.$$

Example 5.5.13. One simple example of an odd function is $f(x) = x$. Notice that, for some positive real number a, the integral $\int_{-a}^{a} f(x)\,dx$ is represented by the picture below.



Notice that the shaded regions to the left and right of the y-axis are equal, but have opposite sign since area under a curve is "negative". So the area under the curve $y = f(x)$ over the interval $[-a, 0]$ is effectively cancels the area over the interval $[0, a]$. In other words,

$$\int_{-a}^{a} x\,dx = 0.$$

The same sort of symmetry applies in general to even and odd function.

Theorem 5.5.14. *Suppose f is continuous on $[-a, a]$.*

1. *If f is even, then* $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

2. *If f is odd, then* $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

Proof. First notice that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx \tag{5.5.2}$$

For the first integral (with bounds 0 and $-a$), let

$$\begin{aligned} u &= -x, \\ du &= -dx. \end{aligned}$$

Our new bounds are

$$\begin{aligned} u(0) &= 0 \\ u(-a) &= -(-a) = a. \end{aligned}$$

Then

$$\begin{aligned} -\int_0^{-a} f(x)\,dx &= -\int_0^a f(-u)(-du) \\ &= \int_0^a f(-u)\,du \\ &= \int_0^a f(-x)\,dx, \qquad \text{(since } u \text{ was a dummy variable)} \end{aligned}$$

so from Equation 5.5.2, we can write

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \tag{5.5.3}$$

1. If f is even, we have $f(-x) = f(x)$, so Equation 5.5.3 yields

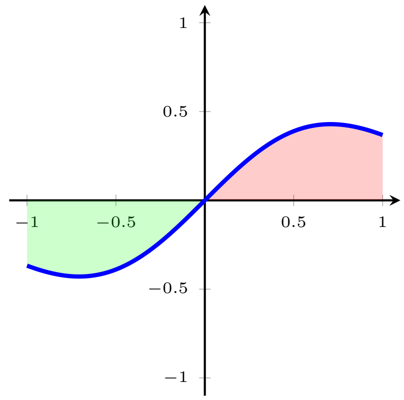
$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

2. If f is odd, we have $f(-x) = -f(x)$, so Equation 5.5.3 yields

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$$

□

Example 5.5.15. Evaluate the integral $\int_{-1}^{1} xe^{-x^2}\,dx$.



We could preform the substitution, but by first graphing the function with our graphing calculator, we can appeal to the geometry and see that the function is odd. Hence

$$\int_{-1}^{1} xe^{-x^2}\,dx = 0.$$

A Appendix: Extra Material

These notes were originally written from an older version of Stewart's *Calculus*. They have since been edited to align with a newer version of his text, and so some subsections have either gone by the wayside or been pushed into a latter chapter of the textbook (and thus into a different semester). I have included them here so that they aren't lost to time, but they were not covered in the course. They are in no particular order.

A.1 Indeterminate Forms and L'Hospital's Rule

A.1.1 0/0 and ∞/∞ Indeterminate Forms

We have seen limits like the following

$$\lim_{x\to\infty}\frac{e^x}{5^x} \quad \text{and} \quad \lim_{x\to 1}\frac{x^2-1}{x-1}.$$

We are unable to simply plug-in the limits as we would end up with "$\frac{\infty}{\infty}$" and "$\frac{0}{0}$", respectively which are both undefined. However, we were able to still find the limits, and what's more, both limits were very different! This reaffirms what we've known - that infinity doesn't quite behave like other real numbers, and division by 0 is equally as ill-behaved.

Definition. Let $f(x)$ and $g(x)$ be functions and consider the limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, then we say this limit is an **indeterminate of type $\frac{0}{0}$.** Similarly, if $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$, then we say that this limit is an **indeterminate form of type $\frac{\infty}{\infty}$.**

For limits of these types, we had various methods of evaluating the limits. For example, in the case of rational functions, we factored and canceled terms. However, such tricks may not work for other "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" indeterminate forms, like

$$\lim_{x\to 0}\frac{\cos x-1}{2^{x+1}-2} \quad \text{or} \quad \lim_{x\to\infty}\frac{\ln x}{x+7}.$$

Now that we have derivatives in our toolbox, we can make use of the following theorem.

Theorem A.1.1. *L'Hospital's Rule Suppose f and g are differentiable functions with $g'(x)\neq 0$ near a (except possibly at a). Suppose also that*

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof. We'll prove the *very* special case where we have an indeterminate form of type $\frac{0}{0}$, $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$; the proof of the theorem in full generality is much more difficult and can be found in the textbook.

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{(since } f(a) = g(a) = 0\text{)} \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \cdot \frac{\frac{1}{x-a}}{\frac{1}{x-a}} \\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} \\
&= \frac{f'(a)}{g'(a)} \\
&= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} && \text{(since } f', g' \text{ are continuous)} \\
&= \lim_{x\to a} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

□

Remark. L'Hospital's rule applies for one-sided limits and limits at infinity as well.

Example A.1.2. Evaluate $\lim\limits_{x\to\infty} \frac{\ln x}{x}$.

We first see that this is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Remark. Notationally, the author likes to use $\overset{LH}{=}$ to indicate when l'Hospital's rule has been applied. This notation is not at all standard.

Example A.1.3. Evaluate $\lim\limits_{t\to 0} \frac{\sin t}{t}$.

We first see that this is an indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's rule.

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{LH}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example A.1.4. Evaluate $\lim\limits_{x\to 1} \frac{\arctan x - \frac{\pi}{4}}{x-1}$.

Recall that $\arctan 1 = \frac{\pi}{4}$, so we indeed have an indeterminate form of type $\frac{0}{0}$, and thus we can apply l'Hospital's rule.

$$\lim_{x\to 1} \frac{\arctan x - \frac{\pi}{4}}{x-1} \overset{LH}{=} \lim_{x\to 1} \frac{\frac{1}{1+x^2}}{1} = \frac{1}{2}.$$

Example A.1.5. Evaluate $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$.

We first see that this is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}} \overset{LH}{=} \lim_{x \to -\infty} \frac{2x}{-e^{-x}}.$$

Once again, this new limit is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule again.

$$\lim_{x \to -\infty} \frac{2x}{-e^{-x}} \overset{LH}{=} \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0.$$

Remark. L'Hospital's rule **can only be applied** to indeterminate forms of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$; for other indeterminate forms (which we'll handle later), the general strategy is to convert them into one of these two indeterminate forms first.

To see why the indeterminate form is an important hypothesis, consider the following (non)example:

Example A.1.6. Evaluate $\lim_{\theta \to \pi^-} \frac{\sin \theta}{1 - \cos \theta}$ by applying l'Hospital's rule blindly. Find the correct limit without l'Hospital's rule.

Blindly applying l'Hospital's to the following limit,

$$\lim_{\theta \to \pi^-} \frac{\sin \theta}{1 - \cos \theta} \overset{LH}{=} \lim_{\theta \to \pi^-} \frac{\cos \theta}{\sin \theta} = -\infty.$$

However, **this is wrong.** To see why, note that $\frac{\sin \theta}{1 - \cos \theta}$ is actually continuous at $\theta = \pi$, so in fact we have

$$\lim_{\theta \to \pi^-} \frac{\sin \theta}{1 - \cos \theta} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0.$$

A.1.2 $0 \cdot \infty$ and ∞-∞ indeterminate forms

Definition. Let f and g be functions and consider the limit

$$\lim_{x \to a} f(x)g(x).$$

If $f(x) \to 0$ and $g(x) \to \pm\infty$ as $x \to a$, then we say that this limit is an **indeterminate form of type $0 \cdot \infty$**.

To deal with limits of this form, the technique is usually to rewrite $f(x)g(x)$ as either $\frac{f(x)}{1/g(x)}$ (resulting in an indeterminate form of type $\frac{0}{0}$) or $\frac{g(x)}{1/g(x)}$ (resulting in an indeterminate form of type $\frac{\infty}{\infty}$) and then applyin l'Hospital's rule.

Example A.1.7. Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

First we see that $e^{-x} \to 0$ and $\sqrt{x} \to \infty$, so indeed we have an indeterminate form of type $0 \cdot \infty$. Rewriting

$$e^{-x}\sqrt{x} = \frac{\sqrt{x}}{e^x},$$

we now have an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} \lim_{x\to\infty} \frac{1}{2e^x\sqrt{x}} = 0.$$

Definition. Let f and g be functions and consider the limit

$$\lim_{x\to a} f(x) - g(x).$$

If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then we say that have an **indeterminate form of type $\infty - \infty$.**

The technique for solving these limits is to convert the difference into a quotient, often by rationalizing or finding a common denominator, and then applying l'Hospital's rule.

Example A.1.8. Evalute $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{1-e^{-x}}\right)$.

By rewriting as a fraction with a common denominator, we have that

$$\frac{1}{x} - \frac{1}{1-e^{-x}} = \frac{1-e^{-x}-x}{x-xe^{-x}},$$

which is indeterminate of type $\frac{0}{0}$. So, applying l'Hospital's rule,

$$\lim_{x\to 0^+} \frac{1-e^{-x}-x}{x-xe^{-x}} \overset{LH}{=} \lim_{x\to 0^+} \frac{e^{-x}-1}{1-e^{-x}+xe^{-x}}.$$

This is again an indeterminate form of type $\frac{0}{0}$. So, applying l'Hospital's rule again,

$$\lim_{x\to 0^+} \frac{e^{-x}-1}{1-e^{-x}+xe^{-x}} \overset{LH}{=} \lim_{x\to 0^+} \frac{-e^{-x}}{e^{-x}+e^{-x}-xe^{-x}} = -\frac{1}{2}.$$

A.1.3 0^0, 1^∞, and ∞^0 indeterminate forms

Definition. Let f and g be functions and consider the limit

$$\lim_{x\to a} [f(x)]^{g(x)}.$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then we say that have an **indeterminate form of type 0^0.**

If $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a$, then we say that have an **indeterminate form of type 1^∞.**

If $f(x) \to \infty$ and $g(x) \to 0$ as $x \to a$, then we say that have an **indeterminate form of type ∞^0.**

There are two equivalent techniques to handle these types of indeterminate forms (and in fac. The first is to take a natural logarithm of the limit. That is, let $L = \lim_{x\to a}[f(x)]^{g(x)}$. Then, since the logarithmic functions are continuous,

$$\ln L = \ln\Big(\lim_{x\to a}[f(x)]^{g(x)}\Big) = \lim_{x\to a}\ln\big([f(x)]^{g(x)}\big) = \lim_{x\to a} g(x)\cdot\ln(f(x)).$$

Once you have solved for this limit on the right (usually by rewriting into "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" and applying L'Hospital's rule), simply "undo" this natural logarithm by raising the function into the exponent with base e; i.e.

$$L = e^{\ln L} = e^{\lim_{x\to a} g(x)\cdot\ln(f(x))}.$$

The other technique is to appeal to the fact that $[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$, that is,

$$\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a} e^{g(x)\ln(f(x))}.$$

Given that exponential functions are equivalent, this just amounts to solving for the limit of $g(x)\ln(f(x))$, and so the technique is basically equivalent to the first one; either should work.

Example A.1.9. Evaluate $\lim_{x\to 0^+} x^x$. Recall that, because the natural log is continuous, we have

$$\lim_{x\to a}\ln(f(x)) = \ln\Big(\lim_{x\to a} f(x)\Big).$$

We see that this limit is an indeterminate form of type 0^0. So, let L be the limit. Taking a natural logarithm of the limit, we get

$$\ln L = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} x\ln x.$$

This is now indeterminate of type $0\cdot\infty$, so we rewrite $x\ln x = \frac{\ln x}{1/x}$ to put change it to an indeterminate form of type $\frac{\infty}{\infty}$, whence we can apply l'Hospital's Rule.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{LH}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

But we're not quite done yet. Notice that we just showed that $\ln L = 0$, but we wanted to find the value of L. From here we deduce that the value of the limit is $L = 1$.

Example A.1.10. Evaluate $\lim_{x\to\infty}\left(1+\dfrac{7}{x}\right)^x$.

We see that this is an indeterminate form of type 1^∞. Just as last time, let L be the value of the limit. Taking a natural logarithm of the limit, we get

$$\ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x\right) = \lim_{x\to\infty}\ln\left(\left(1+\frac{7}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{7}{x}\right).$$

As $x\to\infty$, $\ln\big(1+\frac{7}{x}\big)\to 0$, so we now have an indeterminate form of type $0\cdot\infty$. Rewriting as $\frac{\ln\left(1+\frac{7}{x}\right)}{1/x}$, we get an indeterminate form of type $\frac{\infty}{\infty}$, and can thus apply l'Hospital's rule.

$$\lim_{x\to\infty} x\ln\left(1+\frac{7}{x}\right) = \lim_{x\to\infty}\frac{\ln\big(1+\frac{7}{x}\big)}{1/x} \overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{1+7/x}\cdot\frac{-7}{x^2}}{-1/x^2} = \lim_{x\to\infty}\frac{-\frac{7}{x^2+7x}}{-1/x^2} = \lim_{x\to\infty}\frac{7x^2}{x^2+7x} = 7.$$

So now we have that $\ln L = 7$, and thus $L = e^7$.

Example A.1.11. Evaluate $\lim_{x\to\infty} x^{1/x}$.

We notice that this is indeterminate of type ∞^0, so can rewrite $x^{1/x} = e^{(1/x)\ln x}$. Since exponential functions are continuous, taking the whole limit amounts to taking a limit of the exponent. So, let $M = \lim_{x\to\infty} \frac{1}{x}\ln x$, which we recognize as an indeterminate form of type $0\cdot\infty$. Rearranging it as $\frac{\ln x}{x}$, which is indeterminate of type $\frac{\infty}{\infty}$, and so we can apply l'Hospital's Rule.

$$M = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

So then our entire limit is

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{(1/x)\ln x} = e^M = e^0 = 1.$$

A.2 Average Value of a Function

Recall that the average value of n numbers, $y_1, \ldots, y_n$ is given by

$$y_{\text{avg}} = (y_1 + y_2 + \cdots + y_n)\,\frac{1}{n}.$$

If these numbers came from a function f, so that $y_1 = f(x_1)$, $y_2 = f(x_2)$, etc., then we would have

$$f_{\text{avg}} = (f(x_1) + f(x_2) + \cdots + f(x_n))\,\frac{1}{n}.$$

Notice that this sum looks like a right Riemann sum with n rectangles over the interval $[0,1]$. Since the x_i may instead come from the interval $[a,b]$, multiplying both sides by $(b-a)$, we get

$$f_{\text{avg}}(b-a) = (f(x_1) + f(x_2) + \cdots + f(x_n))\left(\frac{b-a}{n}\right) = \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right)$$

$$\Rightarrow f_{\text{avg}} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right).$$

So, as we let $n\to\infty$, we get the following definition

Definition. The **average value of f on** $[a,b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example A.2.1. Find the average value of $f(x) = 25 - x^2$ on $[0,2]$

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{2-0}\int_0^2 25 - x^2\,dx \\ &= \frac{1}{2}\left[25x - \frac{1}{3}x^3\right]_0^2 \\ &= \frac{1}{2}\left[\left(25(2) - \frac{1}{3}(2)^3\right) - \left(25(0) - \frac{1}{3}(0)^3\right)\right] \\ &= \frac{71}{3}. \end{aligned}$$

Theorem A.2.2 (Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ so that*

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof. Let $F(x) = \int_a^x f(t)\,dt$. Note that $F(a) = 0$. Applying the FTC and the Mean Value Theorem, there exists a c in $(0, 2)$ so that

$$\begin{aligned} f(c) &= F'(c) \\ &= \frac{F(b) - F(a)}{b-a} \\ &= \frac{1}{b-a}F(b) \\ &= \frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned}$$

□

Example A.2.3. With the same function and interval as in Example A.2.1. Find the value of c satisfying the Mean Value Theorem for Integrals.

We're solving for $f(c) = \frac{71}{3}$, so

$$\begin{aligned} 25 - x^2 &= \frac{71}{3} \\ c^2 &= \frac{4}{3} \\ c &= \sqrt{\frac{4}{3}} \approx 0.667. \end{aligned}$$